Vertex cuts, ends and group splittings

Boundaries, Graz 2009

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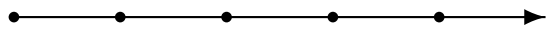
joint with Martin J. Dunwoody (Univ. of Southampton)

Contents

ends of graphs

graph $X = (VX, EX)$ ray R 

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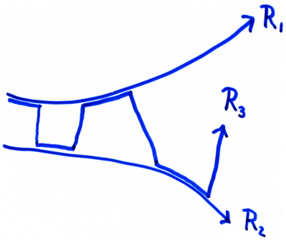

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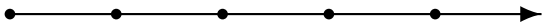
graph $X = (VX, EX)$ ray R 

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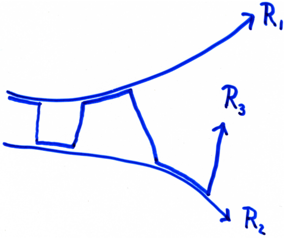

Equivalence classes are the ends of X.

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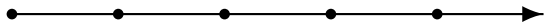
graph $X = (VX, EX)$ ray R 

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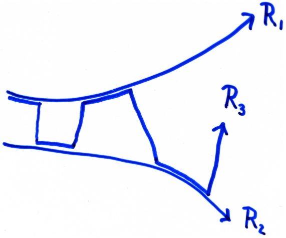

Equivalence classes are the ends of X.

The set of ends ΩX is totally disconnected.

Stallings' structure theorem

The number of ends of $X = \mathrm{Cay}(G, S)$ does not depend on S if $|S| < \infty$.

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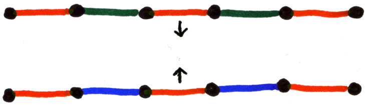
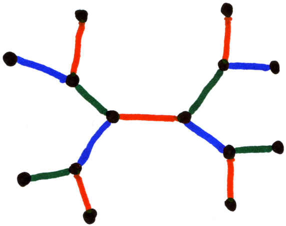
Theorem (Stallings' structure theorem, 1969)

G finitely generated. Then

G has more than one end $\iff$ G splits over finite subgroup

free amalgamated products





$$D_\infty *_{(\mathbb{Z}/2\mathbb{Z})} D_\infty = \langle a, b, c \mid a^2, b^2, c^2 \rangle = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$$

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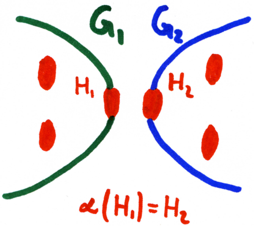

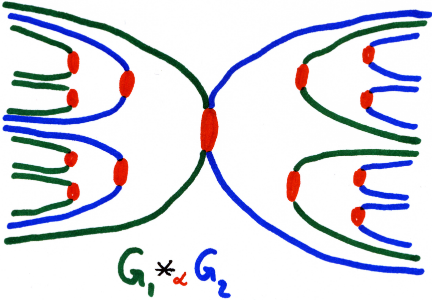

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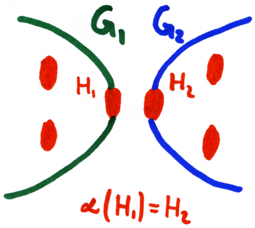

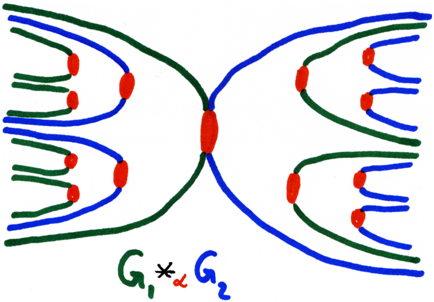

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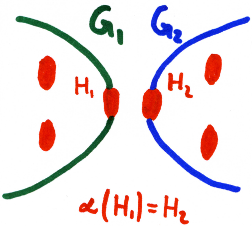

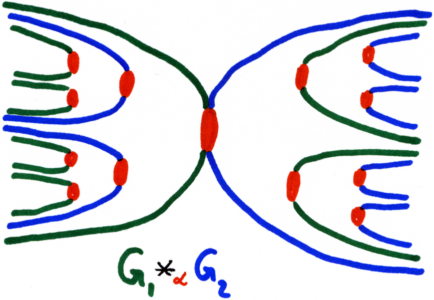

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Only in the case of a standard presentation (as in the picture above) it is clear what the splitting is.

an (almost) trivial example

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answer: infinitely many vertex ends (definition above), one edge end.

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cuts

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Consider cuts as "large, connected sets with small boundary", whatever "large" and "small" may mean.

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Set $\mathcal{E} = \mathcal{C} \cup \mathcal{C}^* = \{C, C^* \mid C \in \mathcal{C}\}$.

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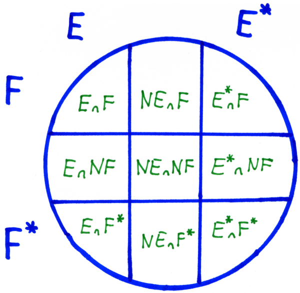

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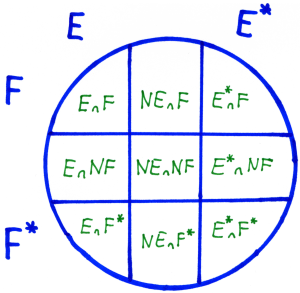

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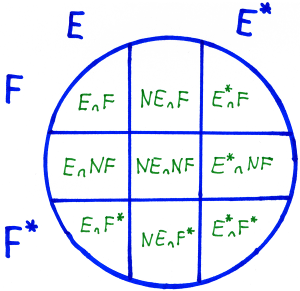
(A3)' If C and D are in $\mathcal{C}$ then $C \setminus ND$ has a component which is an element of $\mathcal{C}$.

diagram nestedness



isolated corner: contains no cut (is small), and adjacent links are empty.

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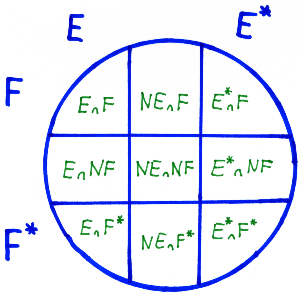

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E and F are nested $\iff$ there is an isolated corner.

examples of cut systems

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$\mathcal{C}$ $=$ connected sets C such that C and C^* contain an end (a ray).

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A subset Y of VX is said to be *k-inseparable* if it has at least $k+1$ elements and if for every set $B \subset VX$ with $|NB| \leq k$, either $Y \subset B \cup NB$ or $Y \subset B^* \cup NB$.

Let κ be the smallest positive integer for which there are sets A, Y_1 and Y_2 such that $|NA| = \kappa$, Y_1 and Y_2 are κ-inseparable, $Y_1 \subset A \cup NA$ and $Y_2 \subset A^* \cup NA$.

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$\mathcal{C} \;=\;$ sets A with the above property and $|NA| = \kappa$.

minimal cut systems

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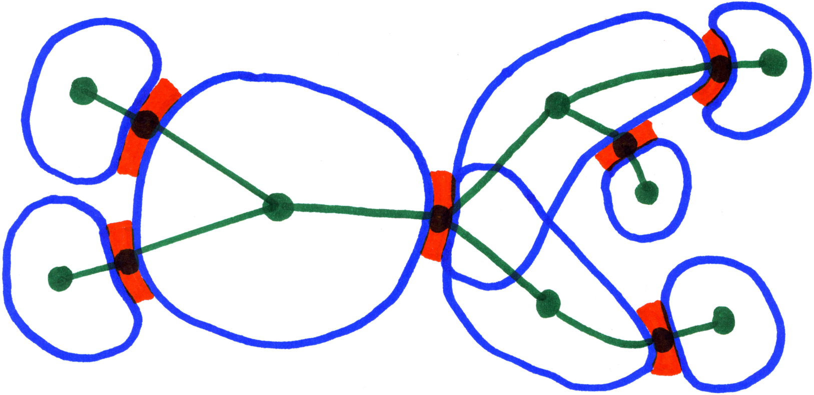
Theorem

Every cut system contains a minimal sub-system.

cuts $\longrightarrow$ tree



$\hat{X}$ and the tree construction

When we replace the boundaries NC of cuts by complete graphs and "cut off" the isolated corners we obtain a connected graph $\hat{X}$.
$X \to \hat{X}, \quad \mathcal{C} \to \hat{\mathcal{C}},$ the structure essentially remains the same.

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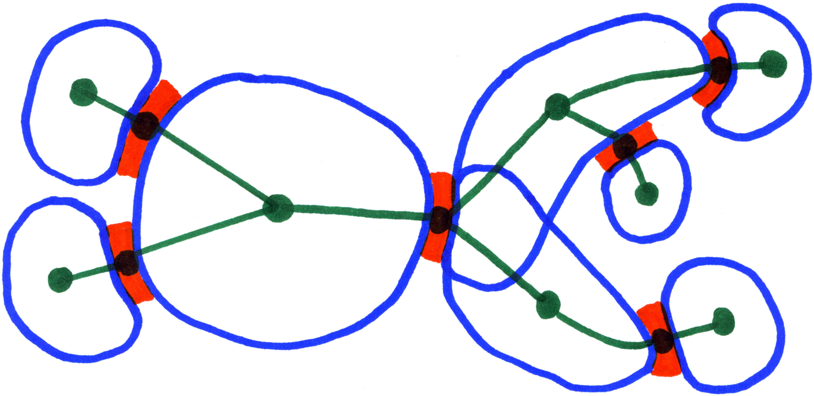
$t(E) = NE$ of $E \in \hat{\mathcal{C}}$ and $o(E)$ is the $\sim$-class which contains E.

cuts → tree



G-cut-systems and G-trees

Main Theorem (Dunwoody, Krön)

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This generalizes edge cuts to vertex cuts, see
Dunwoody "Cutting up graphs" (1982),
Dicks and Dunwoody "Groups acting on graphs" (1989).

Application 1: New proof of Stallings' theorem

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When applying the arguments in the new proof to the classical context we obtain a relatively simple proof of Stallings' theorem on ends of groups.

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