Linear drift on regular covers

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Abstract

We discuss properties of the linear drift of the Brownian motion on a regular Riemannian cover. It is compared with other asymptotic quantities.

Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold: $\widetilde{M}$ is a Riemannian manifold and there is a discrete group Γ of isometries of $\widetilde{M}$ such that the quotient $M = \Gamma \setminus \widetilde{M}$ is a compact manifold. The quotient metric makes M a compact Riemannian manifold.

We consider the Laplacean $\widetilde{\Delta}, \Delta$ on $\widetilde{M}$ and M, the corresponding heat kernels $\widetilde{p}(t, \widetilde{x}, \widetilde{y}), p(t, x, y)$ and the associated Brownian motions $\widetilde{X}, X$. The following quantities were introduced by Guivarc'h ([G]) and Kaimanovich ([K1]), respectively, as almost everywhere limits on the space of trajectories of the Brownian motion $\widetilde{X}$:

- the linear drift $\ell := \lim_{t\to\infty} \frac{1}{t} d_{\widetilde{M}}(\widetilde{X}_0, \widetilde{X}_t)$,
- the entropy $h := \lim_{t\to\infty} -\frac{1}{t} \ln \widetilde{p}(t, \widetilde{X}_0, \widetilde{X}_t)$.

1 Statement

In this talk, we announce the following result (Work in Progress):

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Theorem 1 *Let $\pi : \widetilde{M} \to M$ be a regular Riemannian cover of a compact manifold. With the above notations, we have:*

$$\ell^2 \ \leq \ h. \tag{1}$$

Comments.

1. Inequality (1) is sharp: if $\widetilde{M}$ is the hyperbolic space $\mathbb{H}^n$, $\ell^2 = h = (n-1)^2$.

2. Similar quantities can be defined for a symmetric random walk on a finitely generated group, where the distance on the group is the word distance. It is known that, given a finite symmetric generating set S, there is a constant $c(G,S)$ such that $\ell^2 \leq c(G,S)h$ (Varopoulos [Va] for a random walk with finite support, Erschler-Karlsson with finite second moment [EK]).

3. In particular, whenever $h = 0$, which is equivalent to the Liouville property of $\widetilde{M}$, then $\ell = 0$, as shown in [KL1].

4. In the case when $\widetilde{M}$ is the universal covering of a compact manifold with negative curvature, inequality (1) is due to V. Kaimanovich ([K1]). Moreover, in that case, there is equality in (1) if, and only if, the manifold M is locally symmetric. What equality entails in the general case is not clear yet.

5. Let v be the volume growth of $\widetilde{M}$

$$v \ := \ \lim_{R \to \infty} \frac{1}{R} \ln Vol(B_{\widetilde{M}}(x_0, R)),$$

where $B_{\widetilde{M}}(x_0, R)$ is the ball of radius R in $\widetilde{M}$ about a given point x_0 and Vol is the Riemannian volume. It holds: $h \leq \ell v$([G]).

Corollary 2 *In the setting of Theorem 1, we have $\ell \leq v$ and $h \leq v^2$. Either equality $\ell = v$, $h = v^2$ implies equality in (1).*

6. Let λ be the bottom of the spectrum of the Laplacean on $\widetilde{M}$: $\lambda := \inf_{f \in C^2_K(\widetilde{M})} \frac{\int_{\widetilde{M}} \|\nabla f\|^2}{\int_{\widetilde{M}} |f|^2}$. Clearly (by considering C^2_K approximations to the functions $e^{-sd(x_0,.)}$ for $s > v/2$), we have $4\lambda \leq v^2$. It can be shown that $4\lambda \leq h$ ([L1]). Therefore,

Corollary 3 *In the setting of Theorem 1, equality $4\lambda = v^2$ implies equality in (1).*

2 Sketch of the proof of Theorem 1

We construct the *horospheric suspension* of the action of Γ on $\widetilde{M}$. Since the space $\widetilde{M}$ is a Riemannian manifold with bounded sectional curvature, it is a *proper* metric space (closed bounded subsets are compact) and we consider the *Busemann compactification* of $\widetilde{M}$. Fix a point $x_0 \in \widetilde{M}$ and define, for $x \in \widetilde{M}$ the function $\xi_x(z)$ on $\widetilde{M}$ by:

$$\xi_x(z) \ = \ d(x,z) - d(x,x_0).$$

The assignment $x \mapsto \xi_x$ is continuous, injective and takes values in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The *Busemann compactification* $\widehat{M}$ of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology. The space $\widehat{M}$ is a compact separable space. The *Busemann boundary* $\partial\widetilde{M} := \widehat{M} \setminus \widetilde{M}$ is made of Lipschitz continuous functions ξ on $\widetilde{M}$ such that $\xi(x_0) = 0$. Elements of $\partial\widetilde{M}$ are called *horofunctions*. Observe that we may extend by continuity the action of Γ from $\widetilde{M}$ to $\widehat{M}$, in such a way that for ξ in $\widehat{M}$ and γ in Γ:

$$\gamma.\xi(z) \ = \ \xi(\gamma^{-1}z) - \xi(\gamma^{-1}(x_0)).$$

We define now the *horospheric suspension* X_M of M as the quotient of the space $\widetilde{M} \times \widehat{M}$ by the diagonal action of Γ. The projection onto the first component in $\widetilde{M} \times \widehat{M}$ factors into a projection from X_M to M so that the fibers are isometric to $\widehat{M}$. It is clear that the space X_M is metric compact. To each point $\xi \in \widehat{M}$ is associated the projection W_ξ of $\widetilde{M} \times \{\xi\}$. As a subgroup of Γ, the stabilizer Γ_ξ of the point ξ acts discretely on $\widetilde{M}$ and the space W_ξ is homeomorphic to the quotient of $\widetilde{M}$ by Γ_ξ. We put on each W_ξ the smooth structure and the metric inherited from $\widetilde{M}$. The manifold W_ξ and its metric vary continuously on X_M. The collection of all $W_\xi, \xi \in \widehat{M}$ form a continuous lamination $\mathcal{W}_M$ with leaves which are manifolds locally modeled on $\widetilde{M}$. In particular, it makes sense to differentiate along the leaves of the lamination and we denote $\Delta^{\mathcal{W}}$ the laminated Laplace operator acting on functions which are smooth along the leaves of the lamination. A probability measure m on X_M is called *harmonic* if, for any function f for which it makes sense, we have:

$$\int \Delta^{\mathcal{W}} f dm \ = \ 0.$$

By [Ga], there exist harmonic measures and the set of harmonic probability measures is a weak* compact set of measures on X_M. Moreover, if m is a harmonic measure, and $\widetilde{m}$ is the Γ invariant measure which extends m on $\widetilde{M} \times \widehat{M}$, then ([Ga]) there is a finite measure ν on $\widehat{M}$, and for ν almost every ξ, a positive harmonic function $k(x, \xi)$ with $k(x_0, \xi) = 1$ such that the measure $\widetilde{m}$ can be written as:

$$\widetilde{m} \; = \; k(x, \xi)(dx \times \nu(d\xi)).$$

The harmonic probability measure m is called ergodic harmonic if it is extremal among harmonic probability measures. In this case, for ν almost all ξ, the following limits exist along almost every trajectory of the Brownian motion (see [K2]):

- the linear drift $\ell(m) := \lim_{t \to \infty} \frac{1}{t} \xi(\widetilde{X}_t)$,
- the tranverse entropy $k(m) := \lim_{t \to \infty} -\frac{1}{t} \ln k(\widetilde{X}_t, \xi)$.

It was proven by Kaimanovich ([K2]) that $0 \leq k(m) \leq h$. The proof of Theorem 1 reduces to the two following results:

Proposition 4 *With the above notations, there exists an ergodic harmonic measure m such that $\ell(m) = \ell$.*

Proposition 5 *For all ergodic harmonic measure m, we have $\ell^2(m) \leq k(m)$, with equality if, and only if, the harmonic functions k_ξ are such that, for almost every ξ, $\log k_\xi$ is proportional to ξ.*

The proof of Proposition 4 is an extension of the proof the Furstenberg formula in [KL2]. Kaimanovich ([K2]) proved Proposition 5 under the hypothesis that the horofunctions are of class C^2 by applying Itô's formula to the function ξ. In the general case, horofunctions are only uniformly 1-Lipschitz, but an integrated form of the same idea yields the same result.

3 Further properties in the case of negative curvature

The case when the manifold M has negative sectional curvature and $\widetilde{M}$ is the universal cover of M is familiar: the Busemann boundary is the geometric boundary. The unit tangent bundle T^1M is homeomorphic to a $\mathcal{W}$-saturated subset of X_M and the lamination $\mathcal{W}$ restricted to T^1M is the stable foliation of the geodesic flow. Using tools from the ergodic theory of Anosov flows, one can show that the process $d_{\widetilde{M}}(\widetilde{X}_0, \widetilde{X}_t)$ has the qualitative properties of a real Brownian motion. In particular:

Theorem 6 *[Central Limit Theorem [L2]] There is a positive number σ such that, as t goes to infinity, the distribution of the variable*

$$\frac{1}{\sigma\sqrt{t}}\left(d_{\widetilde{M}}(\widetilde{X}_0, \widetilde{X}_t) - t\ell\right) \tag{2}$$

converges towards the normal distribution.

Set $G(\widetilde{x}, \widetilde{y})$ for the Green function on $\widetilde{M}$, $G(\widetilde{x}, \widetilde{y}) = \int_0^\infty \widetilde{p}(t, \widetilde{x}, \widetilde{y})dt$.

Theorem 7 *[Renewal Theorem [L3]] Consider, for $R > 0$, the integral $U(R)$ of the Green function $G(\widetilde{x}, \widetilde{y})$ on the sphere $S(\widetilde{x}, R)$ of center $\widetilde{x}$ and radius R. Then, as R goes to infinity,*

$$U(R) \to \frac{1}{\ell}.$$

Theorem 7 appears as a renewal theorem for the process $d_{\widetilde{M}}(\widetilde{x}, \widetilde{X}_t)$, since $U(R) = \int_0^\infty \varphi(t, R)dt$, where $\varphi(t, R) := \int_{S(\widetilde{x}, R)} \widetilde{p}(t, \widetilde{x}, \widetilde{y})d\widetilde{y}$ is the occupation density at R of the process $d_{\widetilde{M}}(\widetilde{x}, \widetilde{X}_t)$.

Theorem 6 does not hold in the general case (see [E] for an example in the case of random walks). A question is whether Theorem 7 holds for all coverings such that the Brownian motion is transient on $\widetilde{M}$.

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