Abelian Sandpile Model and Self-Similar Groups

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G a finite connected graph.

$\theta: V(G) \to N$ configuration of the game on G:

$\theta(v)$ is the number of chips at vertex v.

Game move : if $\theta(v) \geq deg(v)$, then v is "unstable" and will be "toppled" (or "fired"):

$\theta'(v) = \theta(v) - deg(v)$; $\theta'(w) = \theta(w) + Adj(v,w)$.

Objective : stabilize an unstable configuration.

The total number of chips is at most $\beta_1(G)$

(Björner, Lovasz, Shor, '91)

or $\Rightarrow$ **any configuration**

There is at least one dissipative vertex v_0 **stabilizes in finite time**

(Dhar, '90)

Abelian property : the resulting stable configuration doesn't depend on the order in which unstable vertices were toppled.

(Dhar, '90; Björner-Lovasz-Shor, '91)

The process.

Once stabilized, we reactivate the game by dropping a new chip at random on $V(G)\setminus\{v_0\}$.

S = {stable configurations}

Operators A_v acting on S, $v \in V(G) \setminus\{v_0\}$:

$$A_v(\theta) = \mathrm{stab}(\theta + \delta_v)$$

Avalanche = sequence of topplings which occur as a result of application of A_v

Let θ be a stable configuration.

$\theta_1 = \theta \quad \rightarrow \quad \theta_1 + \delta_{x_1}$; x_1 a vertex picked at random

$\downarrow$ **an "****avalanche****" occurs**

$\theta_2 = \text{stab}(\theta_1 + \delta_{x_1}) \rightarrow \theta_2 + \delta_{x_2}$; x_2 picked at random

$\downarrow$ avalanche

$\text{stab}(\theta_2 + \delta_{x_2})$ etc.

Markov chain on the set S of stable configurations

{Stable config-s} = Recurrent Ů Transient

Recurrent = Critical ,

i.e., configurations without forbidden sub-configuration.

Sandpile cellular automaton.

Let U a subset of V(G). A sub-configuration $\theta|_U$

Is forbidden if for $\forall$ v in U, $\theta(v) < deg_U(v)$.

The set of critical (or allowed) configurations is stable

under the dynamics of the model.

"The critical configurations are those which are

barely stable." (P. Bak)

Burning algorithm: describes recurrent configurations

and establishes bijection with spanning trees. (Dhar, '91)

Operators A_v : Stable $\to$ Stable

$$\theta \to \text{stab}(\theta + \delta_v)$$

generate a commutative semi-group.

It becomes a (finite, Abelian) group when restricted to Recurrent configurations.

$\text{Crit}(G) = < A_v|_{\text{Reccurent}} \; ; v \in V(G) \setminus \{v_0\} >$

$|\text{Crit}(G)| = \text{complexity}(G)$. (Corollary of the burning algorithm).

Crit(G) $\approx$ {Recurrent configurations, $\oplus$}. θ, θ' recurrent;

$$\theta \oplus \theta' := \text{stab}(\theta + \theta')$$

Stationary distribution for the Markov chain is the uniform distribution μ on Crit(G).

Jacobian of a finite graph

G a finite connected graph.

d: $C^0(G,R) \to C^1(G,R)$ boundary operator

$e=(u,v) \in$ Edges(G), $df(e) := f(v)-f(u)$.

Laplacian on G = d^*d

$d\,d^*$ = Laplacian on the 1-forms

$Harm^1(G,R)$ = space of flows. Dimension = $\beta_1(G)$.

$\Lambda := Harm^1(G,R) \cap C^1(G,R)$ **lattice of integral flows.**

Bacher-de la Harpe-N. '97

$\Lambda^{\#}/\Lambda$ is a finite abelian group called Jacobian of G.

Biggs '98 : **Crit(G) ≈ Jac(G)**

More on Jacobians of finite graphs :

Bacher - de la Harpe - N. (‘97) :

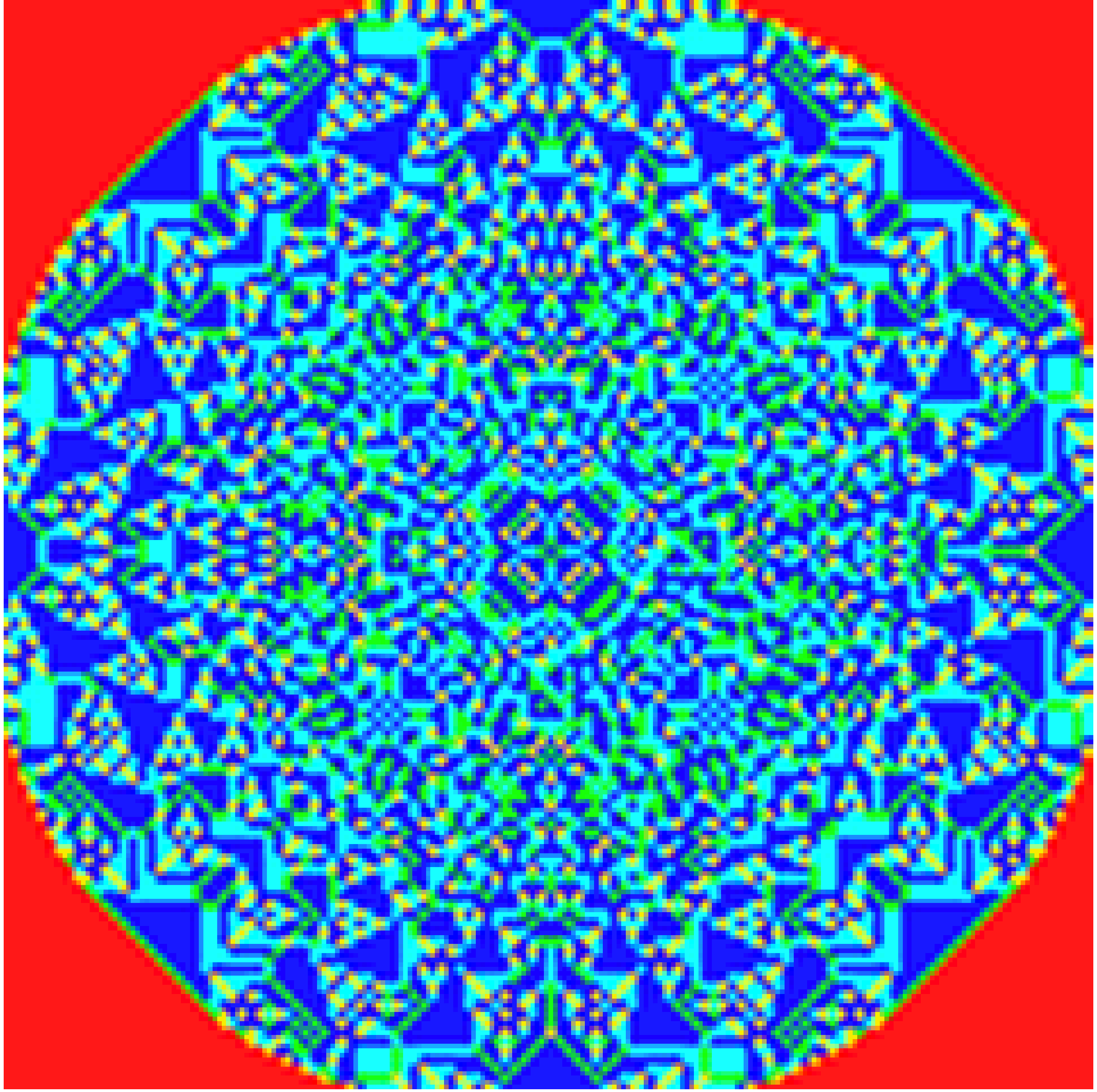
- Abel-Jacobi map $J_v: V(G) \rightarrow Jac(G)$. It is harmonic.
- Jac (G) ≈ Picard group of G ;
- Universal property of Jac(G) .

Baker – Norine (‘07) : Riemann - Roch Theorem.

Similar construction: Kotani-Sunada (’01) : use Jacobian(G) to study random walks on abelian covers of G (crystal lattices).

Caporaso - Viviani (’09) : Torelli Theorem.

Mikhalkin - Zharkov (’08) : Jacobian of a tropical curve.



40000 grains dropped at the point (0,0) in the square 120x120

Bak, Tang, Wiesenfeld ('87):

Self-Organized Criticality

Γ_n = (n×n) - square ; $\Gamma_n \rightarrow Z^2$ as $n \to \infty$.

$\Gamma_n \subset \Gamma_{n+1}$; $\partial\Gamma_n$ = dissipative vertices.

ASM on Z^2 is critical, i.e., spatial correlations in the large volume decay with a power law.

E.g., study asymptotical distributions of avalanches.

Avalanche = the sequence of topplings triggered by adding a grain at some vertex $v_n \in V(\Gamma_n)$ on a critical configuration θ, taken uniformly at random over $Crit(\Gamma_n)$.

Criticality :

Prob (avalanches of large size) ~ size $^{-\alpha}$, as $n \to \infty$

What is known?

Γ = Bethe lattice (infinite binary tree) :

P [Mass(Aval) = k] ~ $k^{-3/2}$, $n \to \infty$ (Dhar).

This is the only example with rigorously proven critical behavior.

Γ = Z^d , d≥2 : **critical behavior is conjectured**, according to phisysists' predictions. **No rigorous proof.**

For d>4 the critical exponent is conjectured to be 3/2.

Γ = Sierpinski gasket : conjecturally critical (numerical simulations)

Γ = Z (or virtually Z): no critical behaviour (Redig, Dhar, Jarai-Lyons)

- Get more examples of critical ASM !

- Open problem :

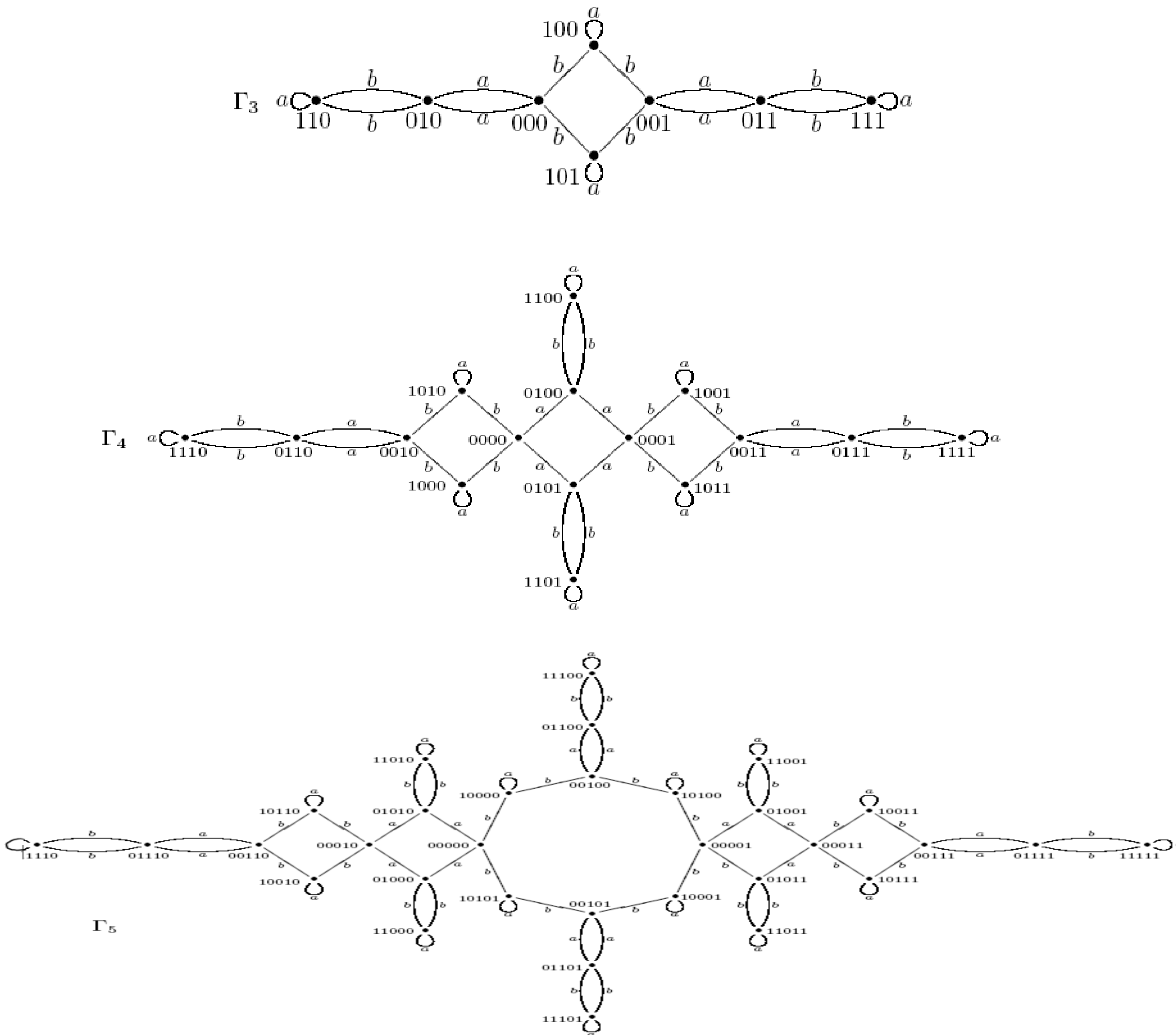
Give a mathematical proof of criticality of ASM on **Z^2**

Find the critical exponent.

Result (Matter – N. '09) :

Uncountably many infinite 4-regular graphs with one end, of quadratic growth, with (rigorously proven) critical ASM.

- Another interesting problem: define ASM in infinite volume. Done for Z^d (Jarai-Redig, Athreya-Jarai).



Γ_3
100
110
010
000
001
011
111
101
Γ_4
1100
1010
0100
1001
1110
0110
0010
0000
0001
0011
0111
1111
1000
0101
1011
1101
Γ_5
11100
01100
11010
11001
10000
00100
10100
10110
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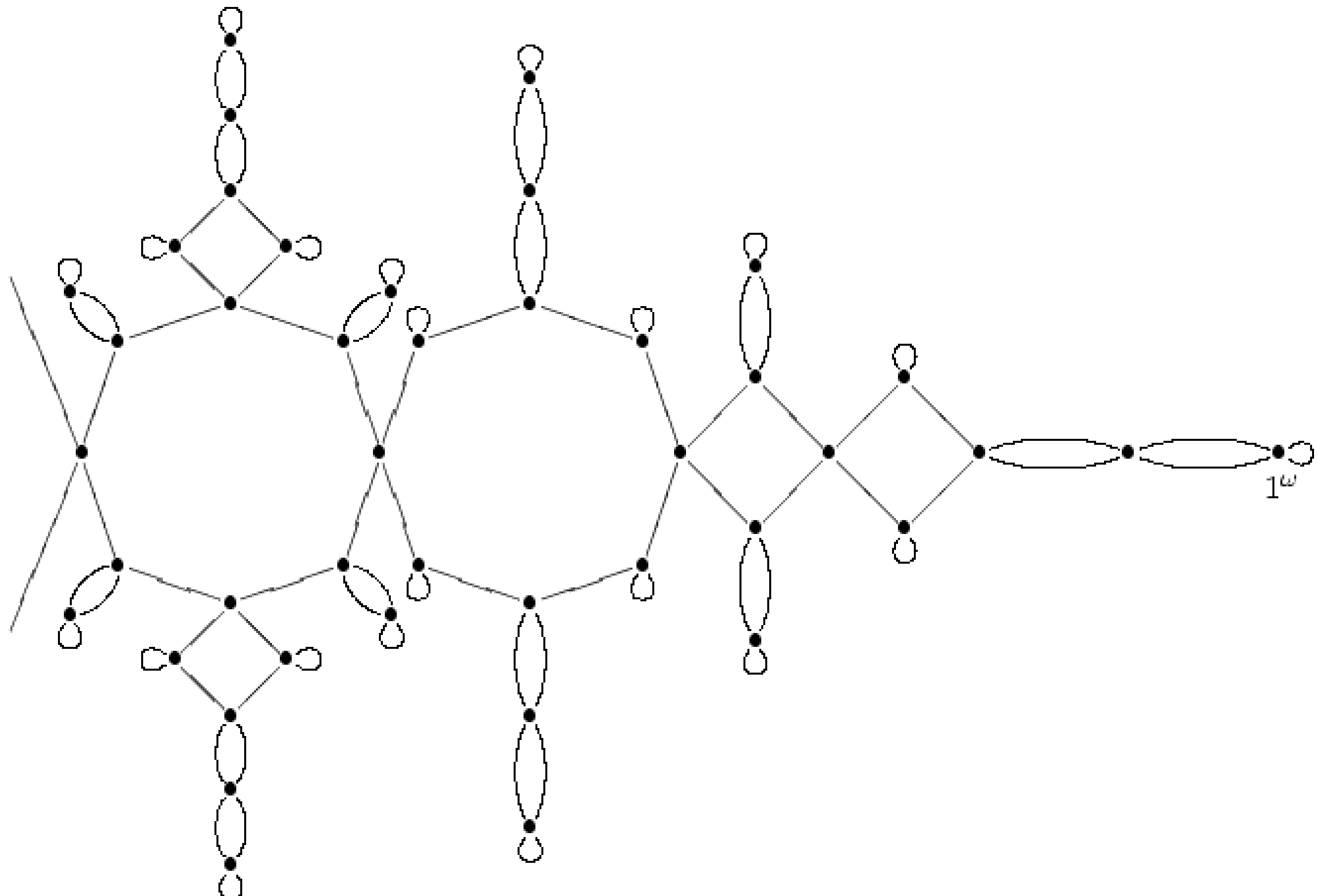
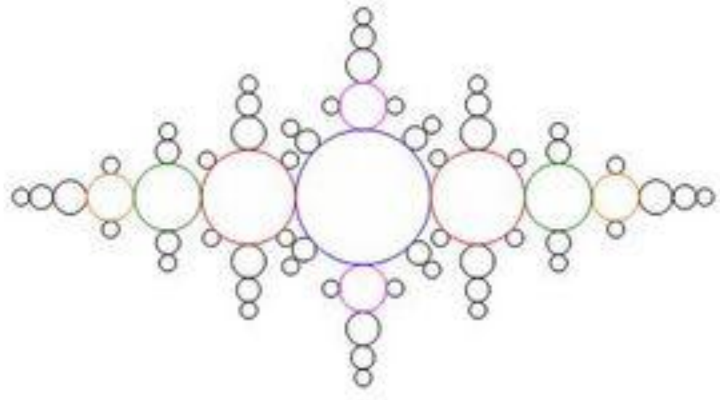


The Schreier graph $(\Gamma, 1^\omega)$.

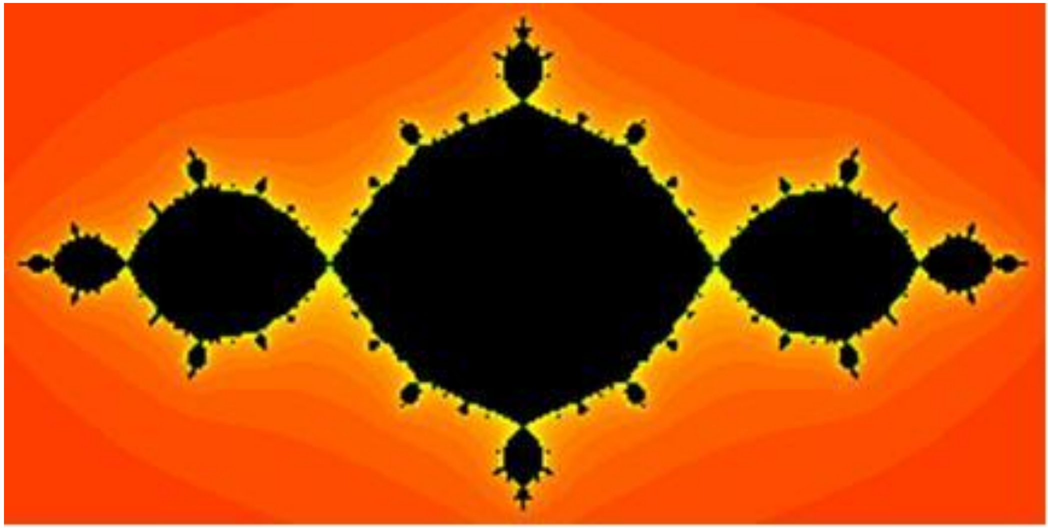
Γ_6



$n \to \infty$

. . .

Scaling limit
called Limit space of G
(Nekrashevych)



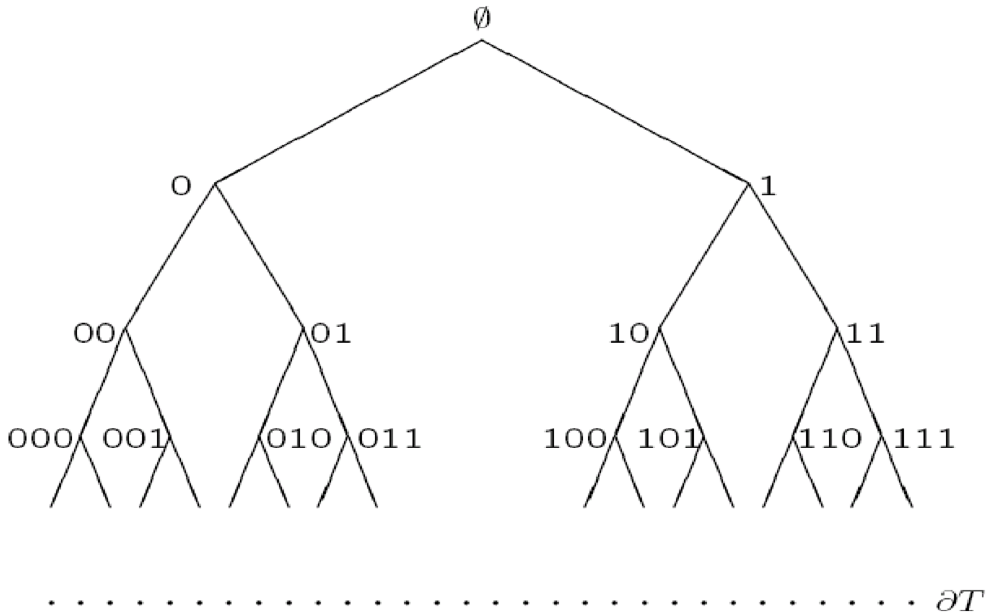
Julia set of z^2-1

Schreier graphs of self-similar groups

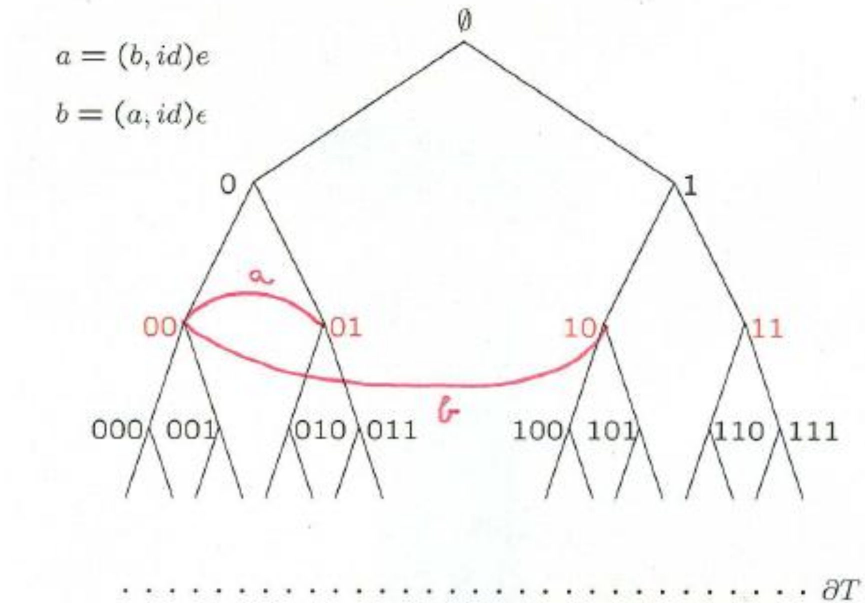
$T=T_d$ - infinite d-ary tree; $V(T_d) = X^*$, $X=\{0,\ldots d-1\}$.

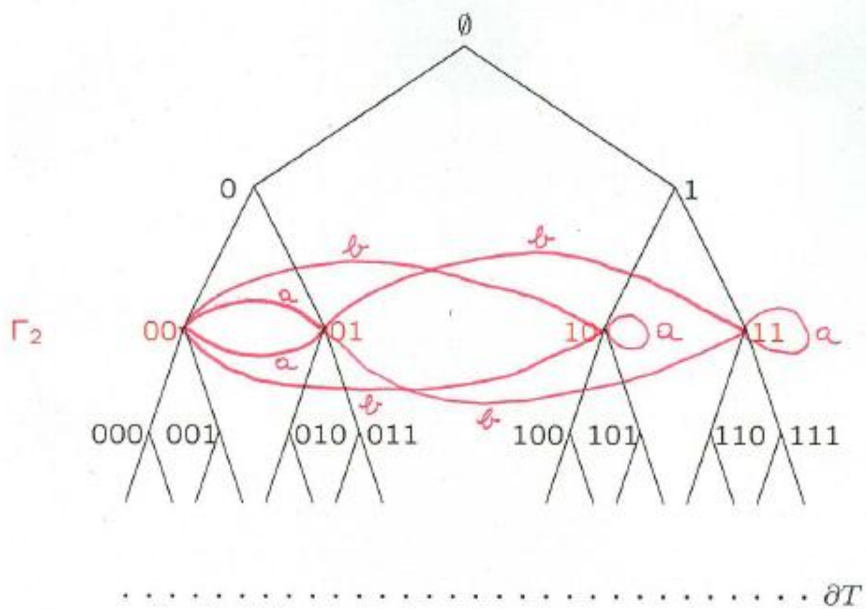
$G < Aut(T)$, G finitely generated.

G acts also on $\partial T = \{ \xi=x_0x_1x_2\ldots\ldots \}$.



The rooted binary tree





G<Aut(T),
transitive on levels.
G = <S>

Γ_n = Sch (G, S, X_n)

Vertices = X_n ,
Edges = {(v,s(v)) | s in S}

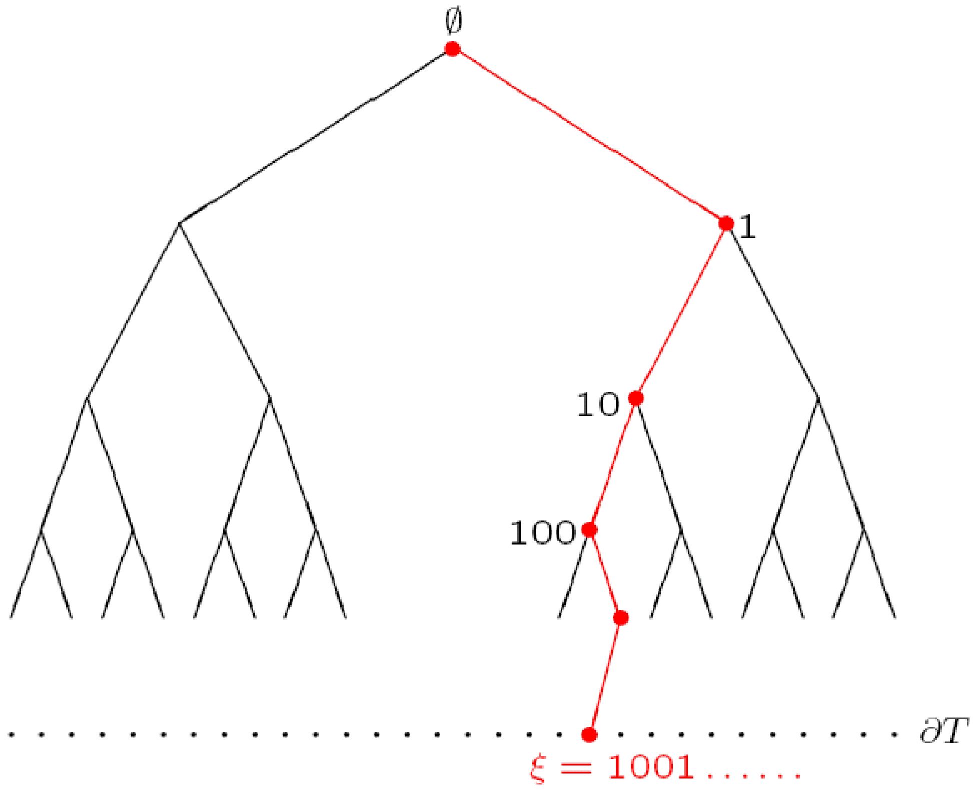
|Vert(Γ_n)| = d^n

Let $\xi = x_0 x_1 x_2 \ldots$ be a boundary point.

Γ_ξ = Sch (G, S, $\mathrm{Stab}_G(\xi)$) infinite (orbital) Schreier graph

$(\Gamma_n, x_0 \ldots x_n) \to (\Gamma_\xi, \xi)$ in the space of rooted graphs.

as $n \to \infty$



Study ASM on Schreier graphs of self-similar groups

Γ_ξ an infinite Schreier graph, Γ_n finite Schreier graphs

$\Gamma_n \to \Gamma_\xi$ as $n \to \infty$. $\xi = x_1 x_2 \ldots$.

θ a critical configuration on Γ_n .

Destabilize it by adding a grain at $x_1 \ldots x_n$.

Study the asymptotics of the triggered avalanches,

under the stationary (uniform) distribution on Crit(Γ_n),

as $n \to \infty$ (in large volume). Try to define a limit process on Γ_ξ.

Key example: Basilica group $B=\langle a,b\rangle$

$a=(b,id)e$, $b=(a,id)\varepsilon$

Use classification of infinite Schreier graphs Γ_ξ as function of boundary point ξ (D'Angeli – Donno – Matter - N. '09).

Recall: For Basilica, Γ_ξ has 1,2 or 4 ends.

E_1 = {rays which give graphs with 1 end}.

$E_1 = \{\xi = x_1y_1x_2y_2\ldots$ both $\{x_i\}$ and $\{y_i\}$ have infinitely many 1's$\}$.

E_1 is subset of ∂T of measure 1.

E_1 is partitioned into uncountably many isomorphism classes, each of measure 0.

Sandpiles on Schreier graphs of Basilica (Matter-N. '09)

Γ_ξ a infinite Schreier graph, Γ_n finite Schreier graphs

$\Gamma_n \to \Gamma_\xi$ as $n \to \infty$. $\xi = x_1 x_2 \ldots$

- Pick a critical configuration uniformly at random;
- destabilize it by adding a grain at $\xi_n = x_1 \ldots x_n$.
- Study the behavior of the triggered avalanche.

Thm . 1) Γ_ξ has 2 or 4 ends $\Rightarrow$ non-critical behavior.

2) For almost every ξ in E_1, avalanches on

$(\Gamma_{\xi n})$ exhibit critical behavior as $n \to \infty$:

$$k^{-4/3} \lesssim P\,[\text{Mass(Aval)} = k] \lesssim k^{-1} .$$

For $\xi = 1111\ldots$, $P[\text{Mass(Aval)} = k] \sim k^{-1}$

To estimate the asymptotics of avalanches use

Matter ('09) : Exact solution of the Abelian Sand-Pile model on "cycle-tree" graphs (cacti).

Γ is a cycle-tree graph if it can be obtained from a tree by replacing edges by cycles.

Remark. Many self-similar groups have Schreier graphs cycle-trees (Nekrashevych)