Random walks on Affine Buildings
(Parkinson, Schapira)

Boundaries, Graz 2/7/09

§1 p-adic Lie groups

$G(\mathbb{F})$ a Chevalley gp over local field $\mathbb{F}$

Example: $G = SL_3$; $\mathbb{F} = \mathbb{F}_q((t))$

Subgroups:

$$\begin{array}{ccc}
G = SL_3(\mathbb{F}_q((t))) & & \\
\cup\!\mid & & \\
K = SL_3(\mathbb{F}_q[[t]]) & \xrightarrow{\theta = ev_{t=0}} & SL_3(\mathbb{F}_q) \\
\cup\!\mid & & \cup\!\mid \\
I = \theta^{-1}(B(\mathbb{F}_q)) & \xrightarrow{\theta} & B(\mathbb{F}_q)
\end{array}$$

$$I = \text{Iwahori subgp} = \left\{ \begin{pmatrix} \mathbb{F}_q[[t]]^\times & & \mathbb{F}_q[[t]] \\ & \ddots & \\ t\mathbb{F}_q[[t]] & & \mathbb{F}_q[[t]]^\times \end{pmatrix} \right\}$$

G/I is the <u>affine flag variety</u>

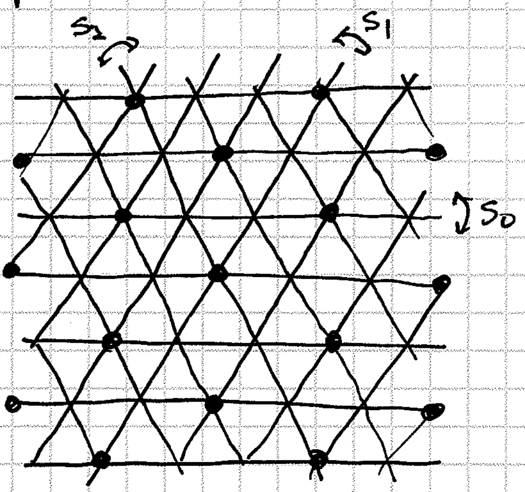
Structure of G/I:

A) Thm (Bruhat/Iwahori decomposition)

$$G = \bigsqcup_{w \in W} IwI$$

where W is the affine Weyl group.

Example: $SL_3(\mathbb{F}_q((t)))$



$$W_0 = \langle s_1, s_2 \rangle \simeq S_3$$

$$W = \langle s_0, s_1, s_2 \rangle = W_0 \ltimes Q$$

B) If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced, then

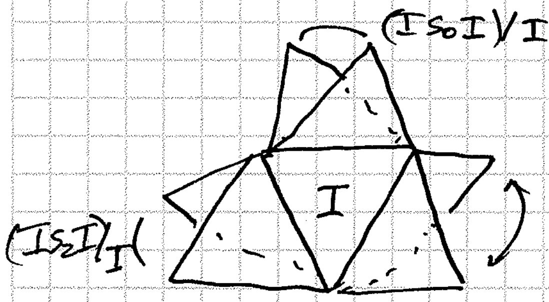
$$(IwI)/I = \{ x_{i_1}(c_1) n_{i_1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell} I \mid c_1, \ldots, c_\ell \in \mathbb{F}_q \}$$

$$\longleftrightarrow \mathbb{F}_q^{\ell}$$

* Complexity arises from fact that usually reduced expression not unique.

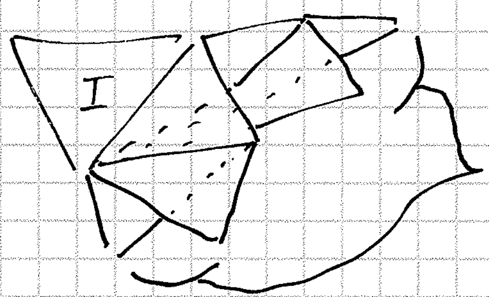
§2 The affine building

The building of G/I is a way of organising G/I into a geometric object that encodes the combinatorics of the Bruhat decomposition and the structure of double cosets.



$(Is_1I)/I = \{x_1(c)n_1I \mid c \in \mathbb{F}_q\} \longleftrightarrow \mathbb{F}_q.$

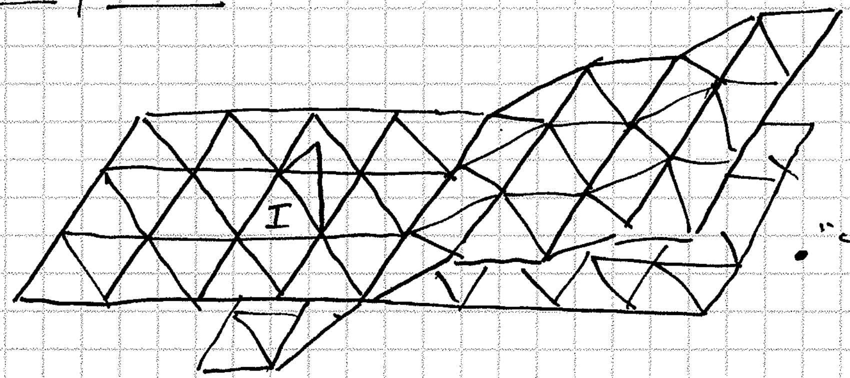
Continue building outwards:



q^2 cosets in Is_1s_2I

$$gI \sim_i hI \iff g^{-1}hI \subseteq Is_iI.$$

Global picture:



- "Apartments"
- "Thick" version of W

The triangles are called chambers: $\mathcal{C} = G/I$

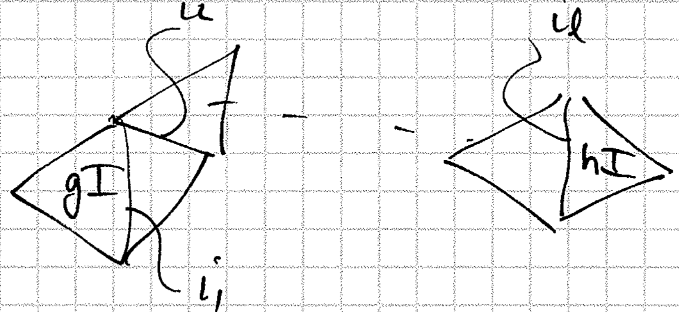
"Distance" between chambers measured by

$$\delta : \mathcal{C} \times \mathcal{C} \longrightarrow W$$

$$\delta(gI, hI) = w \iff g^{-1}hI \subseteq IwI$$

This amounts to the following: If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced, then

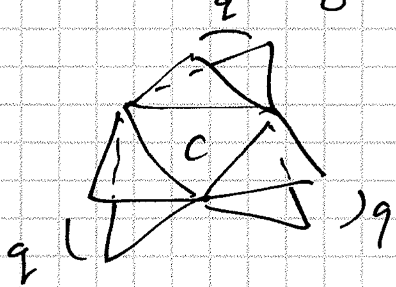
$\delta(gI, hI) = w \iff$



§3. Random walks:

State space $= \mathcal{C}$ (set of chambers)

- Simple nearest neighbors:



$$p(c,d) = \begin{cases} \frac{1}{3q} & \text{if } c \sim d,\ c \neq d \\ 0 & \text{otherwise} \end{cases}$$

- More generally: A walk $P = (p(c,d))_{c,d \in \mathcal{C}}$ is <u>radial</u> if

$$p(c,d) = p(c',d') \text{ whenever } \delta(c,d) = \delta(c',d').$$

Aim: prove a local limit thm for radial walks on affine buildings (loc. finite + regular) ie, asymptotics of $p^{(n)}(c,d)$ as $n \to \infty$.

<u>Remark</u>: Gives local limit theorem for bi-I-invariant probability measures on the p-adic group G.

Conjecture. If $P = (p(c,d))$ is radial then

$$p^{(n)}(c,d) \sim C_{\delta(c,d)}\, \rho^n n^{-k}$$

where $k = \frac{(\text{dimension}) + (\#\text{roots})}{2}$

ρ = largest eigenvalue of $\pi(P)$

where π = principal series rep of affine Hecke algebra with central character $t = 1$.

Remarks: 1) Checked for $\tilde{A}_1$, $\tilde{A}_2$ $(\tilde{B}_2)$

2) Walks on vertices $(= G/K)$ are easier and studied (Cartwright, Woess, P)
k agrees with the k for these walks

3) Local limit thm for real Lie gps proved by Bougerol. Same k

4) Harish-Chandra's Lefschetz principal (any thing true for real reductive gps should be true for p-adic gps).

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§4. Technique

Let $T_w = (p_w(c,d))$ be transition operator for the walk

$$p_w(c,d) = \begin{cases} \frac{1}{q^{\ell(w)}} & \text{if } \delta(c,d) = w \\ 0 & \text{if otherwise} \end{cases}$$

Then $P = (p(x,y))$ is radial $\Longleftrightarrow$

$$P = \sum_{w \in W} a_w T_w \quad \text{with} \quad a_w \geq 0, \ \sum a_w = 1.$$

Let $\mathcal{H} = \mathbb{C}\text{-span}\{T_w \mid w \in W\}$

<u>Lemma</u> If $w \in W$ then

$$T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) = \ell(w)+1 \\ q^{-1} T_{ws_i} + (1-q^{-1}) T_w & \text{if } \ell(ws_i) = \ell(w)-1 \end{cases}$$

<u>Cor:</u> $\mathcal{H}$ is an affine Hecke algebra.

<u>Rem:</u> "Thick" version of the group algebra of W

- Then $(p^{(n)}(c,d)) = P^n = \sum_{w \in W} a_w^{(n)} T_w$

and $\boxed{p^{(n)}(c,c) = a_1^{(n)} = \tau(P^n)}$, where

$\tau : \mathcal{H} \longrightarrow \mathbb{C}$ is defined by linearly extending

$\tau(T_w) = \delta_{w,1}$.

- Then $(h_1, h_2) = \tau(h_1^* h_2)$ defines an inner product on $\mathcal{H}$ (where $T_w^* = T_{w^{-1}}$)

- Then $\mathcal{H} \circlearrowleft \mathcal{H}$ gives operator norm

- Let $\overline{\mathcal{H}}$ = ~~closure~~ completion of $\mathcal{H}$

* $\overline{\mathcal{H}}$ is a C^* algebra

* τ = "state" of $\overline{\mathcal{H}}$.

General results imply: $\exists!$ positive Borel probability measure on $\operatorname{spec}(\overline{\mathcal{H}})$ s.t.

$$\boxed{\tau(h) = \int_{\operatorname{spec}(\overline{\mathcal{H}})} \chi_\pi(h)\, d\mu(\pi)}$$

where $\operatorname{spec}(\overline{\mathcal{H}})$ = irreducible $*$-reps of $\mathcal{H}$ that extend to $\overline{\mathcal{H}}$.

Opdam (2003); 115 page paper

- computes μ
- $\mathrm{spec}(\bar{\mathcal{H}})$ = extensions of irreducible tempered representations induced from cuspidal reps of parabolic subalgebras of $\mathcal{H}$.

* Parallels Harish-Chandra's work on p-adic gps.

<u>Then</u>:

$$P^{(n)}(c,c) = \tau(P^n)$$

$$= \int_{\mathrm{spec}(\bar{\mathcal{H}})} \chi_\pi(P^n)\, d\mu(\pi)$$

$$= \int_{\mathrm{spec}(\bar{\mathcal{H}})} \mathrm{Tr}(\pi(P^n))\, d\mu(\pi)$$

$$= \int_{\mathrm{spec}(\bar{\mathcal{H}})} \mathrm{Tr}(\pi(P)^n)\, d\mu(\pi)$$

6×6 matrix in SL_3 case.

Now one can extract asymptotics. Show that leading behaviour comes from the principal series rep with character 1.