Resistance analysis of infinite networks

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Joint work with Palle E. T. Jorgensen

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Definition (Electrical resistance network (G, c))

A *network* (G, c) is a simple, connected graph $G = \{G^0, G^1\}$ with vertices G^0 and edges G^1.

The edges G^1 are determined by a weight function called *conductance*:

$$x \sim y \text{ iff } 0 < c_{xy} < \infty, \quad \text{for } x, y \in G^0.$$

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The conductance $c : G^0 \times G^0 \to [0, \infty)$ satisfies

$c(x) := \sum_{y \sim x} c_{xy} < \infty$, for all $x \in G^0$, and

$c_{xy} = c_{yx}$ for all $x, y \in G^0$.

Conductance is the reciprocal of the resistance. (Hence the name ERN.)

The energy form $\mathcal{E}$

Definition ((Dirichlet) energy of a function $u : G^0 \to \mathbb{C}$)

$$\mathcal{E}(u) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} |u(x) - u(y)|^2, \quad \operatorname{dom} \mathcal{E} = \{u : \mathcal{E}(u) < \infty\}.$$

$c_{xy} = 0$ unless $x \sim y$; only pairs for which $x \sim y$.
"$\frac{1}{2}$" indicates each edge is counted only once.

For $f : \mathbb{R} \to \mathbb{R}$, the continuous analogue is $\mathcal{E}(f) := \int |f'|^2 \, dx$.

Note: $\operatorname{Ker} \mathcal{E} = \{\textit{constant functions}\}$.

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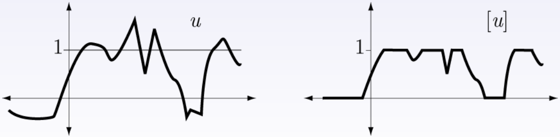
Definition (Energy form $\mathcal{E}$ on functions $u, v \in \operatorname{dom} \mathcal{E}$)

$$\mathcal{E}(u,v) := \frac{1}{2} \sum_{x,y \in G^0} c_{xy} (\overline{u}(x) - \overline{u}(y))(v(x) - v(y))$$

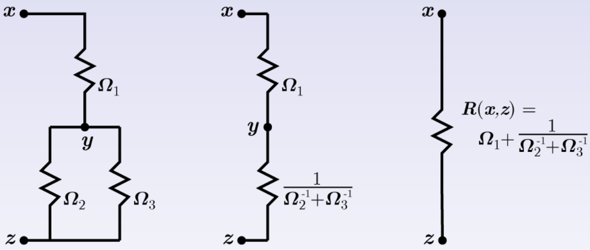
(Polarization) $\mathcal{E}(u,v) = \frac{1}{4}[\mathcal{E}(u+v) - \mathcal{E}(u-v)]$. ($u, v$ $\mathbb{R}$-valued)

(Markov property) $\mathcal{E}([u]) \leq \mathcal{E}(u)$, where $[u]$ is any contraction of u.



Definition (Effective resistance)

The *effective resistance* $R(x, y)$ is the voltage drop between x and y when one unit of current is passed from x to y.



Series addition of resistors: $R = R_1 + R_2$.
Parallel addition of resistors: $R = (R_1^{-1} + R_2^{-1})^{-1}$.

Definition (Effective resistance)

The *effective resistance* $R(x,y)$ is the voltage drop between x and y when one unit of current is passed from x to y.

Theorem: $R(x,y)$ is a metric on G.

Currents may not be unique on infinite networks, so choose one of

$$R^W(x,y) = (\min\{\mathcal{E}(u) : |u(x) - u(y)| = 1\})^{-1}$$
$$R^F(x,y) = \min\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v)\}.$$

These are equal when $\{u \in \operatorname{dom} \mathcal{E} : \Delta u = 0\} = \{constants\}$.

Two roads to the energy space $\mathcal{H}_\mathcal{E}$

(1) Direct construction.

Define $\mathcal{H}_\mathcal{E} := \operatorname{dom} \mathcal{E} / \mathbb{C}\mathbf{1}, \qquad \langle u, v \rangle_\mathcal{E} := \mathcal{E}(u, v).$

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(2) von Neumann embedding.

The metric $R = R^F$ is negative semidefinite.
Therefore, (G, R) embeds into a Hilbert space (von Neumann):
$x \mapsto v_x$ with $R(x, y) = \|v_x - v_y\|_\mathcal{E}^2$.
Up to unitary equivalence, this Hilbert space is $\mathcal{H}_\mathcal{E}$.

Two roads to the energy kernel

(1) Fix $o \in G^0$ (origin). Define $L_x : \mathcal{H}_\mathcal{E} \to \mathbb{C}$ by $L_x u := u(x) - u(o)$.
L_x is continuous on $\mathcal{H}_\mathcal{E}$, so $L_x u = \langle v_x, u \rangle_\mathcal{E}$ for some $v_x \in \mathcal{H}_\mathcal{E}$.

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(2) von Neumann embedding: $x \mapsto v_x$ with $R(x,y) = \|v_x - v_y\|_\mathcal{E}^2$.

Both yield the same set of embedded vertices $\{v_x\}_{x \in G}$.

Definition: The collection $\{v_x\}_{x \in G^0}$ is the *energy kernel*.

Theorem: $\{v_x\}_{x \in G^0}$ is a *reproducing kernel*:

$$\langle v_x, u \rangle_\mathcal{E} = u(x) - u(o) \text{ for all } u \in \mathcal{H}_\mathcal{E}.$$

Choosing representatives with $u(o) = 0$ gives the (discrete) Kuramochi kernel.

Definition ((Network) Laplacian Δ)

A linear difference operator; weighted average of neighbouring values.

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)).$$

If the operator c is multiplication by $c(x) := \sum_{y \sim x} c_{xy}$, then

$$\Delta = c(\mathbb{I} - \mathbf{P}),$$

where $\mathbf{P}$ is the *(probabilistic) transition operator*

$$(\mathbf{P}v)(x) := \frac{1}{c(x)} \sum_{y \sim x} c_{xy} v(y).$$

(stochastically renormalized adjacency matrix)

The structure of $\mathcal{H}_\mathcal{E}$

Definition: $u \in \mathcal{H}_\mathcal{E}$ is a *dipole* iff $\Delta u = \delta_x - \delta_y$ for some $x, y \in G^0$.

Definition: $w \in \mathcal{H}_\mathcal{E}$ is a *monopole* iff $\Delta w = \delta_x$ for some $x \in G^0$.

Definition: $h \in \mathcal{H}_\mathcal{E}$ is *harmonic* iff $\Delta h(x) = 0$ for each $x \in G^0$.

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Theorem: v_x is a dipole: $\Delta v_x = \delta_x - \delta_o$.

Theorem: $\mathcal{H}_\mathcal{E}$ contains monopoles iff the network is transient.

Theorem: If $\mathcal{H}_\mathcal{E}$ contains nonconstant harmonics, the network is transient.

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Theorem: $\mathcal{H}_\mathcal{E} = \mathcal{F}in \oplus \mathcal{H}arm$.

$\mathcal{H}arm := \{h \in \mathcal{H}_\mathcal{E} : \Delta h(x) = 0, \forall x \in G^0\}$, and
$\mathcal{F}in := [\{f \in \mathcal{H}_\mathcal{E} : f(x) = k,$ for all but finitely many $x \in G^0\}]_\mathcal{E}$.

Let w_o be the unique energy-minimizing monopole at the origin. Define $w_x^v := v_x + w_o$ and $w_x^f := f_x + w_o$, where $f_x = P_{\mathcal{F}in} v_x$.

Definition: Let $\mathcal{M} := \text{span}\{v_x\}_{x \in G^0} + \text{span}\{w_x^v, w_x^f\}_{x \in G^0}$.

$\mathcal{M}$ always contains: $\text{span}\{v_x\}, \text{span}\{f_x\}, \text{span}\{h_x\}, \quad h_x = P_{\mathcal{H}arm} v_x$.

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Theorem (Discrete Gauss-Green Formula)
For $u \in \mathcal{H}_{\mathcal{E}}$ and $v \in \mathcal{M}$, $\langle u, v \rangle_{\mathcal{E}} = \sum_{G^0} u \Delta v + \sum_{\operatorname{bd} G} u \frac{\partial v}{\partial \mathrm{n}}$.

$$\int_U \nabla \varphi \cdot \nabla \psi \, dV = -\int_U \varphi \Delta \psi \, dV + \int_{\partial U} \varphi \frac{\partial}{\partial \mathrm{n}} \psi \, dS$$

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Let $G_k \subseteq G_{k+1}$, $G = \bigcup G_k$, as before.

$$\operatorname{bd} G_k := \{x \in G_k : \exists y \in G_k^\complement, y \sim x\}$$

$$\frac{\partial v}{\partial \mathbb{n}}(x) := \sum_{y \in G_k} c_{xy}(v(x) - v(y)), \qquad x \in \operatorname{bd} G_k$$

Think: $\frac{\partial v}{\partial \mathbb{n}}(x) = \Delta\big|_{G_k}(x)$.

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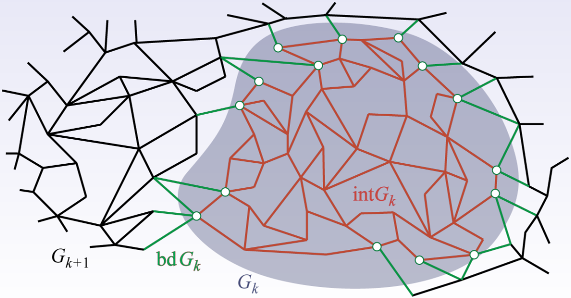

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$$\sum_{\operatorname{bd} G} u \frac{\partial v}{\partial \mathbb{n}} := \lim_{k \to \infty} \sum_{x \in \operatorname{bd} G_k} u(x) \frac{\partial v}{\partial \mathbb{n}}(x).$$

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Theorem: The network is transient if and only if the boundary term is nonvanishing.

I.e., nonzero boundary sum when applied to $\langle u, v\rangle_\mathcal{E}$, for every representative of u except the one specified by $u(x) = \langle u, w_x^v\rangle_\mathcal{E}$.

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Theorem: The network is transient if and only if $(\text{ran}\, \Delta_\mathcal{M}^*)^{c\ell} = \mathcal{F}in$.

$\Delta_\mathcal{M}$ is the closure of Δ when taken to have dense domain $\mathcal{M}$.

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Theorem [Boundary representation]: For all $u \in \mathcal{H}arm$,

$$u(x) = \sum_{\text{bd}\, G} u \frac{\partial h_x}{\partial \mathbb{n}} + u(o).$$

Proof: Note that $u(x) - u(o) = \langle v_x, u\rangle_\mathcal{E} = \overline{\langle u, h_x\rangle_\mathcal{E}} = \sum_{\text{bd}\, G} u \frac{\partial h_x}{\partial \mathbb{n}}$.

$$u(x) = \sum_{\operatorname{bd} G} u \tfrac{\partial h_x}{\partial \mathbb{n}} + u(o) \quad \text{vs.} \quad u(x) = \int_{\partial\Omega} u(y) k(x, dy), \; y \in \partial\Omega.$$

Goals:

- A measure space $\operatorname{bd} G$ and a measure $\mathbb{P}$ on it.
- An extension of $u, h_x \in \mathcal{H}arm$ to elements $\xi \in \operatorname{bd} G$.
- A kernel $\mathbb{k}(x, d\xi) := h_x(\xi) d\mathbb{P}(\xi)$ on $G^0 \times \operatorname{bd} G$.
- An integral representation $u(x) = \int_{\operatorname{bd} G} u(\xi) \mathbb{k}(x, d\xi) + u(o)$.
- A concrete realization of $\xi \in \operatorname{bd} G$.

Problem: $\mathcal{H}_\mathcal{E}$ is too small to support $\mathbb{P}$.

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Solution: construct a Gel'fand triple for $\mathcal{H}_\mathcal{E}$.

$$\mathcal{S} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'.$$

- $\mathcal{S}$ is dense in $\mathcal{H}_\mathcal{E}$ with respect to $\mathcal{E}$.
- $\mathcal{S}$ has another, strictly finer, "test function" topology.
- $\mathcal{S}'$ is the dual of $\mathcal{S}$ with respect to the test function topology.

Think: $\mathcal{S} = \{\text{test functions}\}$ and $\mathcal{S}' = \{\text{distributions}\}$.

The boundary will be some suitable subspace of $\mathcal{S}'$.

Constructing the space of test functions

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Let $\Delta\!\!\!\!*_{\mathcal{M}}^p := (\Delta\!\!\!\!*_{\mathcal{M}} \Delta\!\!\!\!*_{\mathcal{M}} \dots \Delta\!\!\!\!*_{\mathcal{M}})$ be the p-fold application of $\Delta\!\!\!\!*_{\mathcal{M}}$.

$$\operatorname{dom}(\Delta\!\!\!\!*_{\mathcal{M}}^p) := \{u : \Delta\!\!\!\!*_{\mathcal{M}}^{p-1} u \in \operatorname{dom}(\Delta\!\!\!\!*_{\mathcal{M}})\}.$$

Define $\mathcal{S} := \operatorname{dom}(\Delta\!\!\!\!*_{\mathcal{M}}^\infty) := \bigcap_{p=1}^\infty \operatorname{dom}(\Delta\!\!\!\!*_{\mathcal{M}}^p)$.

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Define $\mathcal{S} := \operatorname{dom}({\Delta\!\!\!\!*}_{\mathcal{M}}^{\infty}) := \bigcap_{p=1}^{\infty} \operatorname{dom}({\Delta\!\!\!\!*}_{\mathcal{M}}^{p})$.

Theorem: $\mathcal{S}$ is dense in $\operatorname{dom} {\Delta\!\!\!\!*}_{\mathcal{M}}$ and ${\Delta\!\!\!\!*}_{\mathcal{M}}(\mathcal{S}) \subseteq \mathcal{S}$, and $\mathcal{S}$ is a Fréchet space with seminorms $\|u\|_p := \|{\Delta\!\!\!\!*}_{\mathcal{M}}^{p} u\|_{\mathcal{E}}$.

Theorem. $\mathcal{S}$ is a dense analytic subspace of $\mathcal{H}_{\mathcal{E}}$ (w.r. $\mathcal{E}$).
(Approximate using $E_n u := \int_{1/n}^{n} E(dt) u$.)

A Gel'fand triple for $\mathcal{H}_\mathcal{E}$

Theorem. $\mathcal{S} \subseteq \mathcal{H}_\mathcal{E} \subseteq \mathcal{S}'$ is a Gel'fand triple.

The energy form extends to a pairing on $\mathcal{S} \times \mathcal{S}'$ defined by

$$\langle u, \xi \rangle_{\mathcal{W}} = \langle \Delta_{\mathcal{M}}^{p} u, \Delta_{\mathcal{M}}^{-p} \xi \rangle_\mathcal{E} = \lim_{n \to \infty} \xi(E_n u).$$

Here $E_n u := \int_{1/n}^{n} E(dt) u$.

Note: $\Delta_{\mathcal{M}}^{-p} \xi$ is the p^{th} primitive (antiderivative), not the p^{th} power of $\Delta_{\mathcal{M}}^{-1}$.

$$\xi \in \mathcal{S}' \iff |\xi(u)| \leq C \|\Delta_{\mathcal{M}}^{p} u\|_\mathcal{E},$$

so $\varphi(\Delta_{\mathcal{M}}^{p} u) := \langle u, \xi \rangle$ is continuous on $\operatorname{span}\{\Delta_{\mathcal{M}}^{p} u : u \in \mathcal{H}_\mathcal{E}\}$.

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Theorem. The Wiener transform $\tilde{u}(\xi) := \langle u, \xi \rangle_\mathcal{W}$ gives an isometric embedding $\mathcal{W} : \mathcal{H}_\mathcal{E} \to L^2(\mathcal{S}'_G, \mathbb{P})$, with

$$\langle u, v \rangle_\mathcal{E} = \int_{\mathcal{S}'} \overline{\tilde{u}} \tilde{v} \, d\mathbb{P}.$$

The proof uses:

Bochner: $R^F(x, y) = \|v_x - v_y\|_\mathcal{E}^2$ is negative semidefinite, so $e^{-\frac{1}{2}\|u-v\|_\mathcal{E}^2}$ is positive definite.

Minlos: {pos. def. fns on $\mathcal{S}$ } $\leftrightarrow$ {Radon prob. meas. on $\mathcal{S}'$}.

Wiener: $\mathcal{W} : v_x \mapsto \langle v_x, \cdot \rangle_\mathcal{W}$ is an isometry, so $\langle \cdot, \cdot \rangle_\mathcal{W}$ extends to a pairing on $\mathcal{H}_\mathcal{E} \times \mathcal{S}'_G$.

Boundary integral representation of $u \in \mathcal{H}arm$

Theorem. For $u \in \mathcal{H}arm$ and $h_x = P_{\mathcal{H}arm} v_x$,

$$u(x) = \int_{\mathcal{S}'} u(\xi) h_x(\xi)\, d\mathbb{P}(\xi) + u(o).$$

Compare to $$u(x) = \sum_{\text{bd}\, G} u \tfrac{\partial h_x}{\partial \mathbb{n}} + u(o) \; = \int_{\text{bd}\, G} u(\xi) \mathbb{k}(x, d\xi) + u(o).$$

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From the Wiener isometry:

$$L^2(\mathcal{S}', \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_\mathcal{E}^{\otimes n} = \mathbb{C}\mathbf{1} \oplus \mathcal{H}_\mathcal{E} \oplus \mathcal{H}_\mathcal{E}^2 \oplus \dots.$$

$\mathcal{H}_\mathcal{E}^{\otimes 0} := \mathbb{C}\mathbf{1}$ for a unit "vacuum" vector $\mathbf{1}$.

$\mathcal{H}_\mathcal{E}^{\otimes n}$ is the n-fold symmetric tensor product of $\mathcal{H}_\mathcal{E}$ with itself.

$u \mapsto \langle u, \cdot \rangle \in \mathcal{H}_\mathcal{E}^1$, $(u, v) \mapsto \langle u, \cdot \rangle \langle v, \cdot \rangle \in \mathcal{H}_\mathcal{E}^2$, etc.

Observe that $\mathbf{1}$ is orthogonal to $\mathcal{F}in$ and $\mathcal{H}arm$, but is not the zero element of $L^2(\mathcal{S}_G', \mathbb{P})$.

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The kernel $\mathbb{k}(x, d\xi) = (\mathbf{1} + h_x)d\mathbb{P}$

Since $u(x) = \int_{\mathcal{S}'} u(\xi)h_x(\xi)\, d\mathbb{P}(\xi) + u(o)$, the obvious choice for the kernel is $h_x d\mathbb{P}$.

$k(x, \cdot) = h_x d\mathbb{P}$ is supported on $\mathcal{S}'/\mathcal{F}in$:
since h_x is harmonic, $h_x(\xi) = \langle h_x, \xi \rangle_{\mathcal{W}} = 0$ for $\varphi \in \mathcal{F}in$.

But then $\int_{\mathcal{S}'} k(x, \cdot)\, d\mathbb{P} = \int_{\mathcal{S}'} \mathbf{1} h_x\, d\mathbb{P} = \langle \mathbf{1}, h_x \rangle_{\mathcal{E}} = 0$.
One expects $\int_{\mathcal{S}'} k(x, \cdot)\, d\mathbb{P} = 1$.

The kernel $\mathbb{k}(x, d\xi) = (\mathbf{1} + h_x)d\mathbb{P}$

Since $u(x) = \int_{\mathcal{S}'} u(\xi)h_x(\xi)\,d\mathbb{P}(\xi) + u(o)$, the obvious choice for the kernel is $h_x d\mathbb{P}$.

$k(x, \cdot) = h_x d\mathbb{P}$ is supported on $\mathcal{S}'/\mathcal{F}in$:
since h_x is harmonic, $h_x(\xi) = \langle h_x, \xi \rangle_{\mathcal{W}} = 0$ for $\varphi \in \mathcal{F}in$.

But then $\int_{\mathcal{S}'} k(x, \cdot)\,d\mathbb{P} = \int_{\mathcal{S}'} \mathbf{1}h_x\,d\mathbb{P} = \langle \mathbf{1}, h_x \rangle_{\mathcal{E}} = 0.$
One expects $\int_{\mathcal{S}'} k(x, \cdot)\,d\mathbb{P} = 1$.

Now $\int_{\mathcal{S}'} \mathbb{k}(x, d\xi)\,d\mathbb{P} = \int_{\mathcal{S}'} \mathbf{1}\,d\mathbb{P} + \int_{\mathcal{S}'} h_x\,d\mathbb{P} = 1$.

It follows that $h_x \geq -1$ $\mathbb{P}$-a.e. on $\mathcal{S}'$.

Also, $\mathbb{k}(x, \cdot) \ll \mathbb{P}$ with Radon-Nikodym derivative $\frac{d\mathbb{k}_x}{d\mathbb{P}} = \mathbf{1} + h_x$.

The boundary $\operatorname{bd} G$

A path is a sequence of vertices $\gamma = (x_0, x_1, \dots)$ with $x_i \sim x_{i-1}$.

Define $\gamma \simeq \gamma'$ iff $\lim_{n\to\infty}(h(x_n) - h(x'_n)) = 0$ for every $h \in \mathcal{H}arm$.

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Let $\beta = [\gamma]$ be such an equivalence class. Define

$$\nu_\beta := \lim_{n\to\infty} \mathbb{k}(x_n, d\mathbb{P}).$$

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Alaoglu's theorem gives a weak-$\star$ limit, so for any $u \in \mathcal{H}arm$,

$$u(x_n) = \int_{S'} u(\mathbf{1} + h_x)\, d\mathbb{P} \xrightarrow{n\to\infty} \int_{S'} u\, d\nu_\beta := u(\beta).$$

So $\operatorname{bd} G$ is the set of all equivalence classes of infinite paths in G, under this equivalence relation.

The boundary $\operatorname{bd} G$

Compare $\mathbb{k}(x_n, d\mathbb{P})$ to an approximate identity in Fourier analysis:

$\int_{\mathcal{S}'} \mathbb{k}(x_n, d\mathbb{P}) = 1$ for each n, and

$\lim_{n\to\infty} \int_{\mathcal{S}'} \mathbb{k}(x_n, d\mathbb{P})$ is a Dirac mass (at β).

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Intuition: on any finite subset G_k, define a probability measure μ_x on $\operatorname{bd} G_k$ by

$\mu_x(y) := \mathbb{P}_x[X_{\tau_{\operatorname{bd} G_k}} = y]$, for all $y \in \operatorname{bd} G_k$.

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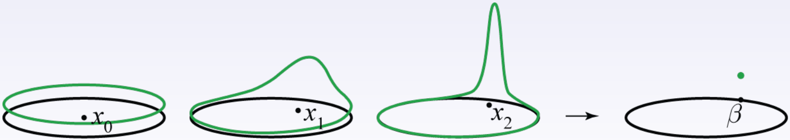
Consider Brownian motion on a disk with such an exit measure.



Definition (Effective resistance)

The *effective resistance* $R(x, y)$ is the voltage drop between x and y when one unit of current is passed from x to y.

$$R^F(x, y) = (v_x - v_y)(x) - (v_x - v_y)(y) \quad (1a)$$

$$= \mathcal{E}(v_x - v_y) \quad (1b)$$

$$= \min\{D(I) : \operatorname{div} I = \delta_x - \delta_y \text{ and } I = \textstyle\sum \xi_\gamma \chi_\gamma\} \quad (1c)$$

$$= (\min\{\mathcal{E}(v) : v(x) = 1, v(y) = 0\})^{-1} + \mathcal{E}(P_{\mathcal{H}arm}(v_x - v_y)) \quad (1d)$$

$$= \inf\{\kappa \geq 0 : |v(x) - v(y)|^2 \leq \kappa \mathcal{E}(v), \forall v \in \operatorname{dom} \mathcal{E}\} \quad (1e)$$

$$= \sup\{|v(x) - v(y)|^2 : \mathcal{E}(v) \leq 1, \forall v \in \operatorname{dom} \mathcal{E}\} \quad (1f)$$

Let $\{G_k\}_{k=1}^\infty$ be an *exhaustion* of G and define $R_{G_k}(a, b)$ as above, except extremize over $u, v : G^0 \to \mathbb{C}$. Then define $R^F(x, y) := \lim_{k\to\infty} R_{G_k}(x, y)$.

Theorem: $x \mapsto v_x$ is an isometric embedding of (G, R^F) into $\mathcal{H}_\mathcal{E}$:

$$R^F(x, y) = \|v_x - v_y\|_\mathcal{E}^2.$$