Boundary convergence of harmonic functions on homogeneous trees and (possibly) buildings

Massimo Picardello

Mathematics Department, University of Roma "Tor Vergata"

Graz, June 29^{th}, 2009

Abstract

We prove admissible convergence to the boundary of functions that are harmonic on subsets of an infinite homogeneous tree with respect to the adjacency operator. The approach is based on a discrete Green formula, suitable estimates for the Green and Poisson kernel and an analogue of the Lusin area function. This is joint work with L. Atanasi (TAMS 2008).
A similar approach might work for harmonic functions on affine buildings. We give an outline of such an approach for the building of PGL_3.

Area function in the half-plane

To start, let us review the continuous case: the Lusin area theorem in the half-plane (or equivalently the disc). Let f be harmonic in the half-plane $\mathbb{H}^+$, for instance. $\omega \in \mathbb{R}$ generic point in the real axis. $\Gamma_\alpha(\omega)$ cone of width α in $\mathbb{H}^+$ with vertex in ω (note: in the hyperbolic distance of $\mathbb{H}^+$ this is a tube with axis $\{\omega + i\tau,\, \tau > 0\}$) Area function:

$$A_\alpha(\omega) := \int_{\Gamma_\alpha(\omega)} \|\nabla f\|^2 \, dx \, dy$$

The Lusin area theorem in the half-plane

Theorem (Lusin area theorem in $\mathbb{H}^+$)

Let $E \subset \mathbb{R}$ measurable, f harmonic on $\mathbb{H}^+$. Then the following are equivalent:

(i) *f is non-tangentially bounded at a.e. $\omega \in E$: $\exists \alpha = \alpha(\omega)$ such that, $E-$almost everywhere, f is bounded on $\Gamma_\alpha(\omega)$;*

(ii) *f has non-tangential limit at almost every $\omega \in E$;*

(iii) *$\exists \alpha = \alpha(\omega)$ such that $A_{\alpha_\omega} f(\omega) < \infty$ for almost every $\omega \in E$;*

(iv) *for every fixed $\alpha \geqslant 0$, $A_\alpha f(\omega) < \infty$ for a.e. in E;*

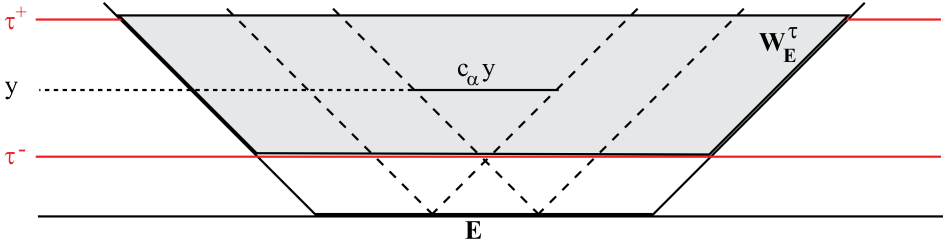
(v) *$\forall$ fixed $\alpha \geqslant 0$, f is bounded on $\Gamma_\alpha(\omega)$ $\omega-$a.e. on E.*

Moreover, $\|A_\alpha f\|_p \sim \|N_\beta f\|_p$, for all $p < \infty$ and all $\alpha, \beta > 0$, where $N_\beta f(\omega) = \sup_{\Gamma_\beta(\omega)} |f|$ is the non-tangential maximal function.

Sketch of proof in the half-plane

Sketch of proof. Up to negligible measure E is a union of intervals. For simplicity let us assume E is an interval and prove $(i) \Rightarrow (iv)$. We are assuming that f is bounded on almost all cones with vertex in E. For the moment, assume more: that $f \in L^{\infty}$. Also, for the time being assume α costant.

Consider the conical extension W_E of E.
Truncate at heights $\tau^- < \tau^+$ (τ^- near the boundary) to get a trapeze W_E^{τ}.



τ^+
W_E^τ
$c_\alpha y$
y
τ^-
E

Now for $\omega \in E$ and $\tau = [\tau^-, \tau^+]$ consider the truncated cone Γ_ω^τ and compute the integral

$$I := \int_E \int_{\Gamma_\omega^\tau} \|\nabla f\|^2 \, dx\, dy\, d\omega$$

When the vertices of two cones are translated along E, their sections at height y overlap by a quantity $w(y)$ proportional to y:

$$w(y) = |\{\omega : y \in \Gamma(\omega)\}| = c_\alpha y$$

But so, w is harmonic on $\mathbb{H}^+$!
Since also f is harmonic,

$$\|\nabla f\|^2 = \|\nabla f\|^2 + 2f \triangle f = \triangle(f^2)$$

so we can now compute the integral I via the Green identities and the Green formula.

$$
\begin{aligned}
I = \int_{W_E^\tau} w \, \triangle(f^2) &= \int_{W_E^\tau} w \, \triangle(f^2) - f^2 \, \triangle w \\
&= \int_{\partial W_E^\tau} w \, \frac{\partial}{\partial n} f^2 - f^2 \, \frac{\partial}{\partial n} w
\end{aligned}
$$

Now remember that $f \in L^\infty$ and observe the following:

- $$\left|\frac{\partial}{\partial n} f^2\right| = 2\left|f \frac{\partial}{\partial n} f\right| \leqslant C|f|^2$$

  by elliptic estimates (or simply because for a bounded harmonic function $\frac{\partial}{\partial n} f \sim f$ by Harnack's inequality...)
- $w = c_\alpha y$ is proportional to the Green function of $\mathbb{H}^+$, hence $\frac{\partial}{\partial n} w \sim w \to$ harmonic measure on boundary. Hence:
- $\int_{\partial W_E^\tau \cap \{y=\tau^+\}} w$ is bounded, and
- independently of τ^-, $\int_{\partial W_E^\tau \cap \{y=\tau^-\}} w \sim \nu(E)$
  (ν is the harmonic measure, i.e., Lebesgue measure on $\mathbb{R}$)

Moreover, the integral of w on the side facets is bounded. So, by letting $\tau^- \to 0^+$ and $\tau^+ \to \infty$, we have

$$\begin{aligned}\lim |I| &= \left| \int_{W_E} w \, \triangle(f^2) \right| \\ &= \left| \int_{\partial W_E^\tau} w \, \frac{\partial}{\partial n} f^2 - f^2 \, \frac{\partial}{\partial n} w \right| \leqslant c \|f\|_\infty^2\end{aligned}$$

hence, for a.a. ω,

$$|A_\alpha(\omega)|^2 \leqslant c\|f\|_\infty^2$$

This proves the theorem if f is bounded. If not, uniformization:

Uniformization

Assume f unbounded, but bounded on almost all cones $\Gamma_\alpha(\omega)$.
Claim: we can choose $E^N \subset E$ such that f is bounded on the conical extension W_{E^N}.
Indeed,

$$E_k := \{\omega : |f| < k \quad \text{on } \Gamma_\alpha(\omega)\}$$
$$E^N := \cup_{k \leqslant N} E_k$$

Since f is bounded on almost each cone, for N large we have

$$\nu(E \setminus E^N) < \varepsilon$$

and on W_{E^N} one has $|f| < N$.

Finally, let us uniformize again to remove the extra assumption that α be costant (this step will not be necessary in a tree). Choose $j \in \mathbb{N}$, let $\beta = \min\{\alpha(\omega), j\}$, $\Gamma_j(\omega) := \Gamma_\beta(\omega)$ and

$$E_{j,k} := \{\omega : |f| < k \quad \text{on } \Gamma_J(\omega)\}$$

Now let

$$E_{M,N} := \cup_{j \leqslant M} \cup_{k \leqslant N} E_{j,k}$$

For every ε, $\nu(E \setminus E_{M,N}) < \varepsilon$ for large M, N, and $|f| < N$ on $E_{M,N}$.

$\square$

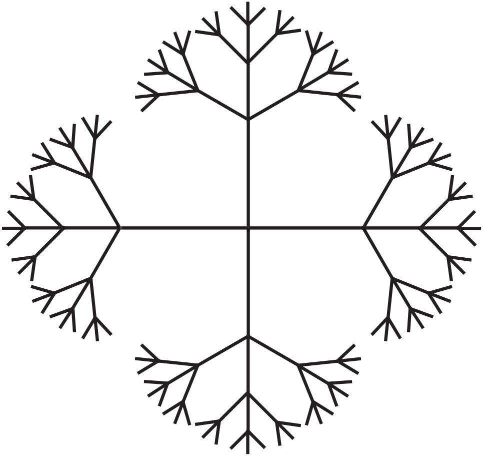
Notation for trees

Definition

We consider an infinite homogeneous tree T: all vertices have the same valency $\mathfrak{h} = q + 1$. We shall also write T for its set of vertices.

Natural distance $d(x, y)$: length of the direct path from x to y.
Write $x \sim y$ if x, y are neighbors: distance 1.
We fix a reference vertex $o \in T$ and call it the origin. The choice of o induces a partial ordering in T: $x \leqslant y$ if x belongs to the geodesic from o to y. Length: $|x| = d(o, x)$.

For any vertex x and $k \leqslant |x|$, x_k is the vertex of length k in the geodesic $[o, x]$.



The boundary of a tree and its topology

Definition (Boundary)

Let Ω be the set of infinite geodesics starting at o. ω_n is the vertex of length n in the geodesic ω. For $x \in T$ the interval $U(x) \subset \Omega$, generated by x, is the set $U(x) = \left\{\omega \in \Omega : x = \omega_{|x|}\right\}$. The sets $U(\omega_n)$, $n \in \mathbb{N}$, form an open basis at $\omega \in \Omega$. Equipped with this topology Ω is compact and totally disconnected.

For every vertex x, the set of vertices $v > x$ form the sector $S(x)$. Call closed sector the set $\overline{S(x)} := S(x) \cup U(x) \subset T \cup \Omega$. The closed sectors induce on $T \cup \Omega$ a compact topology.

Adjacency and harmonic measure

On the vertices of T we consider the adjacency operator A and its normalization, the isotropic nearest neighbor transition operator $P = \frac{1}{\mathfrak{h}} A$.

Transition probabilities $p(u, v) = 1/\mathfrak{h} = 1/(q+1)$ if $u \sim v$

Exponential growth $\Rightarrow$ probability of moving forwards larger than probability of moving backwards $\Rightarrow$ transience.

The sets $\{U(x) : |x| = n\}$ generate a $\sigma-$algebra on Ω. On this $\sigma-$algebra:

Definition (Harmonic measure)

$$\begin{aligned}\nu(U(x)) \equiv \nu_o(U(x)) &:= \frac{1}{\text{number of vertices of length } |x|} \\ &= \frac{1}{\mathfrak{h}(\mathfrak{h}-1)^{|x|-1}} = \frac{1}{(q+1)q^{|x|-1}} = \Pr[X_\infty \in U(x)]\,.\end{aligned}$$

Isotropic Laplacian and harmonic functions

Definition (Laplace operator)

The Laplace operator is $\Delta = P - \mathbb{I}$.

Definition (Harmonic functions)

$f : T \longrightarrow \mathbb{R}$ is harmonic at $x \in T$ if $\Delta f = 0$ that is, if

$Pf(x) \equiv \sum_{y \sim x} p(x,y)\, f(y) =$ Average of f on neighbors $= f(x)$.

Gradient

For $\sigma \in \Lambda$ denote by $b(\sigma)$ the beginning vertex of σ and by $e(\sigma)$ the ending vertex: $\sigma = [b(\sigma), e(\sigma)]$.

Definition (Gradient)

For any function $f : T \to \mathbb{R}$, the gradient $\nabla f : \Lambda \to \mathbb{R}$ is

$$\nabla f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

For $x \in T$,

$$\|\nabla f(x)\|^2 \equiv \sum_{y \sim x} p([x,y]) |\nabla f([x,y])|^2 .$$

Area function on trees

$\forall x \in T$, $\omega \in \Omega$, set $d(x,\omega) = \min_{j\in\mathbb{N}} d(x,\omega_j)$.

Definition

Let $\alpha \geqslant 0$ be an integer. The tube $\Gamma_\alpha(\omega)$ around the geodesic $\omega \in \Omega$ is

$$\Gamma_\alpha(\omega) = \{x \in T : d(x,\omega) \leqslant \alpha\}.$$

Definition (Area function)

$$A_\alpha f(\omega) = \left(\sum_{x\in\Gamma_\alpha(\omega)} \|\nabla f(x)\|^2 \right)^{\frac{1}{2}}.$$

Observe that, if $f \in L^1(T)$, $A_\alpha f(\omega) < \infty$ for every α, ω.

Definition (Non-tangential limits and boundedness)

A function f on T has non-tangential limit at $\omega \in \Omega$ if, for every integer $\alpha \geqslant 0$, $\lim f(x)$ exists as $|x| \to \infty$ and $x \in \Gamma_\alpha(\omega)$.
We say that f has non-tangential limit up to width β if the above limit exists for all $0 \leqslant \alpha \leqslant \beta$.

A function f on T is non-tangentially bounded at $\omega \in \Omega$ if, for some $M > 0$, one has $|f(x)| \leqslant M$ for $x \in \Gamma_0(\omega)$.

Definition (*Conical* - better, tubular - extension of a boundary set)

The "cone" (actually, tube) $W_\alpha(E)$ over a measurable subset E of Ω is

$$W_\alpha(E) = \cup_{\omega\in E}\Gamma_\alpha(\omega).$$

For any integer $s > 0$ let $W^s_\alpha(E) = \cup_{\omega\in E}\Gamma_\alpha(\omega) \cap [s;\infty)$.

The main goal of this paper is the following extension of the Lusin area theorem to non-homogeneous trees. This theorem has been proved for homogeneous trees By L. Atanasi and this speaker (TAMS 2008).

Lusin theorem for trees

Main Theorem (The Lusin Area Theorem)

Let E be a measurable subset of Ω and f a harmonic function on T. Then the following are equivalent:

(i) *f is non-tangentially bounded at almost every $\omega \in E$;*

(ii) *f has non-tangential limit at almost every $\omega \in E$;*

(iii) *$A_0 f(\omega) < \infty$ for almost every $\omega \in E$;*

(iv) *for every fixed $\alpha \geqslant 0$, $A_\alpha f(\omega) < \infty$ for almost every $\omega \in E$.*

The same statement holds if f is harmonic on a connected subset of T whose boundary contains E, or more precisely on some tube $W_\beta(E)$, provided that $\alpha \leqslant \beta$ in (*iv*) and, in (*ii*), f is assumed to be non-tangentially bounded up to width β.

Observe that the definition of non-tangential boundedness is equivalent to say that, for some non-negative integer $\alpha = \alpha(\omega)$, f is bounded on a tube of width α around the geodesic $\Gamma_0(\omega)$ (by a different constant depending on α), and, in this discrete setting, condition (*iii*) in the theorem is equivalent to the more familiar statement that for almost every ω there is some integer α such that $A_\alpha f$ is finite at ω.

Green kernel

Definition (Green kernel)

The Green kernel $G(u, v)$ is the expected number of visits to v of the random walk starting at u:

$$G(u, v) = \sum_{n=0}^{\infty} P^n(u, v) . \tag{1}$$

It satisfies $PG = G - \mathbb{I}$, that is, $\sum_y p(x, y)G(y, z) = G(x, z) - \delta_x(z)$.

Given two positive functions f and g, we write $f \approx g$ if $f < Cg$ and $g < Cf$ for some constant C.

Proposition

For $x \in T$

$$\nu(U(x)) \approx G(o, x) .$$

Poisson kernel

Definition (Poisson kernel)

For every x, $v \in T$ the Martin kernel $K(x,v)$ s defined as

$$K(x,v) \equiv \frac{G(x,v)}{G(o,v)} \approx \frac{d\nu_x}{d\nu_o}(U(v))\,.$$

For every $x \in T$, $\omega \in \Omega$ the Poisson kernel $K(x,\omega)$ is defined as

$$K(x,\omega) \equiv \lim_{v \to \infty} \frac{G(x,v)}{G(o,v)}\,.$$

For every $\omega \in \Omega$, $K(\cdot,\omega)$ is harmonic on T, and integral over Ω with integral 1. It is an approximate identity on Ω:

Poisson integral representation

Proposition (Explicit expression of Poisson kernel)

If $v = v_o(x,\omega)$ join of rays from o to ω and from x to ω, then

- *horocycle index $h(x,\omega) :=$ number of edges from o to $v-$ number of edges from x to v;*
- $K(x,\omega)(= K_o(x,\omega)) = q^{h(x,\omega)}$.

The Poisson integral of a function h in $L^1(\Omega)$ is defined by

$$\mathbf{K}h(x) = \int_\Omega h(\omega) K(x,\omega) d\nu \, .$$

The Green formula

Definition (Boundary of a finite set)

The *boundary* ∂Q of a subset $Q \subset T$ is
$\partial Q = \{\sigma \in \Lambda : b(\sigma) \in Q, e(\sigma) \notin Q\}$.

The Green formula, well known in the continuous setup, has been extended to the discrete context of trees by F. Di Biase and this speaker (Zeitschrift, 1995).

Proposition (The Green formula)

$\forall f$ and h functions on T, $Q \subset T$ finite,

$$\sum_{Q} (h\Delta f - f\Delta h) = (q+1)^{-1} \sum_{\partial Q} (h \circ b \nabla f - f \circ b \nabla h)\,.$$

The Green identity

Proposition (The Green identity)

$$\Delta(f^2)(x) = \|\nabla f(x)\|^2 + 2f(x)\Delta f(x).$$

Sketch of a part of the proof

Let us see a sketch of the same part of the proof that we gave for the half-plane.

Up to a perfectly analogous uniformization procedure, we need to show that boundedness of f on cones implies $A_\alpha f < \infty$ for some α: since $A_0 f \leqslant A_\alpha f$ this is equivalent to show:

Theorem (Lusin theorem, $(i) \Rightarrow (iii)$)

$$f \in L^\infty(T) \Rightarrow \|Af\|_\infty < C\|f\|_\infty$$

Sketch of proof. o reference vertex, y^- predecessor of $y \in T$, $(y \neq o)$.

$$|A_0 f(\omega)|^2 = \sum_{y \text{ vertex in } \omega} |f(y) - f(y^-)|^2$$

$K \subset \mathrm{Aut}(T)$ stability subgroup of o, compact. K can be regarded as a group od automorphisms of Ω.

ν $K-$invariant measure on Ω.

Choose reference point $\overline{\omega} \in \Omega$. Then

$$\begin{aligned}
\int_\Omega |A_0 f(\omega)|^2 \, d\nu &= \int_K \sum_{y \in k\overline{\omega}} |f(y) - f(y^-)|^2 \, dk \\
&= \sum_{n>0} \sum_{|y|=n} |f(y) - f(y^-)|^2 \; |\{k \in K : \; k\overline{\omega}_n = y\}| \\
&:= \sum_{n>0} \sum_{|y|=n} |f(y) - f(y^-)|^2 \, w(n)
\end{aligned}$$

$K_n :=$ stabilizer of $\overline{\omega}_n$ in K. The group K_n fixes the chain $o \to \overline{\omega}_1 \to \overline{\omega}_2 \to \cdots \to \overline{\omega}_n$.

But K acts transitively: so

$$w(n) = [K : K_n] = \frac{1}{\{|y| = n\}} = \frac{1}{(q+1)q^{n-1}}\,.$$

Now, this w is exactly the Green function of the homogeneous tree with singularity at o: $Pw = w$ off o.
Indeed, the average of w at any vertex $y \neq o$ is again w:

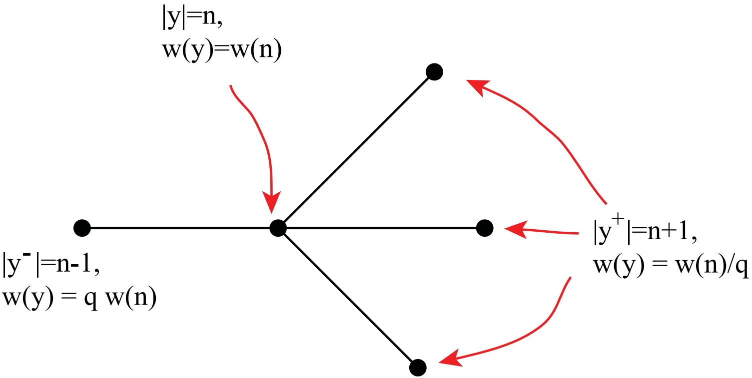


Figure: Harmonicity away from 0 of the isotropic Green function w: $w(y^-) + \sum_{y^+} w(y^+) = qw(y) + qw(y)/q = (q+1)w(y)$

So w is harmonic away from o.

The rest of the proof is now exactly as in the continuous case, via the discrete Green formula and Green identities.

p-adic fields

We outline here a possible approach to the Lusin area theorem on affine buildings of rank 3 (homogeneous trees are affine buildings of rank 2).

p a prime.
Field $\mathcal{F} := \mathbb{Q}_p$: $x \in \mathbb{Q}_p \leftrightarrow x = a_j p^j + a_{j+1} p^{j+1} + a_{j+2} p^{j+2} + \dots$, $a_n \in \mathbb{N}$, $0 \leqslant a_n \leqslant p-1$. The terms grow in $\mathbb{Q}$, but decay geometrically in a suitable $\mathbb{Q}_p$ norm.

$p-$adic valuation: $v(x) = j$

$p-$adic norm: $|x| = p^{-j}$
properties:

- $|xy| = |x||y|$
- $|x+y| = \max\{|x|, |y|\}$ non-archimedean

Ring of integers, principal ideal, lattices

Let us look at the algebraic structure of $\mathbb{Q}_p$. Ring of integers:

$$\mathcal{O} = \{x \in \mathbb{Q}_p : \, |x| \leqslant 1\} = \left\{ x = \sum_{j \geqslant 0} a_j p^j \right\} = \{x : \, v(x) \geqslant 0\}$$

Units: $\mathcal{O}^\times = \{|x| = 1\}$

Maximal ideal:

there is one maximal ideal $\mathcal{P} = \{x : \, v(x) > 0\} = \{x : \, |x| \leqslant \frac{1}{p}\}$

It is principal: $\mathcal{P} = \pi\mathcal{O}$ for some $\pi \in \mathbb{Q}_p$ (choose for instance $\pi = p$)

Lattices: $\mathcal{O}-$modules spanning the vector space $V = \mathbb{Q}_p^n$.

We shall take $n = 3$ from now on.

Lattice= span of a basis $\{v_1, v_2, v_3\}$ over $\mathcal{O}$: we write

$$L = (v_1, v_2, v_3)\,.$$

Origin of the space of lattices: $L_0 = (e_1, e_2, e_3)$, canonical basis.

Stability groups

The ring $M_3(\mathcal{F})$ of 3 by 3 matrices over $\mathcal{F}$ acts on triples of vectors, but not on bases, so it does not map lattices to lattices. $GL_3(\mathcal{F})$ acts on bases.

Proposition

$GL_3(\mathcal{F})$ acts on lattices. The stability subgroup at L_0 is $GL_3(\mathcal{O})$. Hence the space of lattices is isomorphic to the quotient $GL_3(\mathcal{F})/GL_3(\mathcal{O})$.

Sketch of proof. If M maps a lattice to a lattice, it maps a basis to a basis, so $M \in GL_3(\mathcal{F})$.

$ML_0 \subset L_0 \Rightarrow M$ has entries in $\mathcal{O}$. It is easy to see that every $L \in GL_3(\mathcal{O})$ maps L_0 to L_0: this amounts to say that $g \in GL_3(\mathcal{O}) \Leftrightarrow g(\mathcal{O}^3) = \mathcal{O}^3$, that is, $gL_0 = L_0$. $\square$

Remark

A lattice can be identified with a basis, hence with a matrix in $GL_3(\mathcal{F})$. The origin is the identity matrix. A class of equivalence of lattices can be identified with a "matrix" in $PGL_3(\mathcal{F})$.

Building of $PGL_3(\mathbb{Q}_p)$: simultaneous diagonalization

Equivalence relation: $L_1 \equiv L_2$ if $\exists \alpha \in \mathcal{F}^x : \alpha L_1 = L_2$.

Hence the space of classes of equivalence of lattices (or building of PGL_3) is $X = PGL_3(\mathcal{F})/PGL_3(\mathcal{O})$.

Adjacency: $[L_1] \sim [L_2]$ if $pL_1 \subset L_2 \subset L_1$.

Remark

This is equivalent to $pL_2 \subset pL_1 \subset L_2$, hence adjacency is symmetric

Proposition

For all lattices L_1, L_2 there exists $g \in GL_3(\mathcal{F})$ such that $g[L_1] = [e_1, e_2, e_3] = [L_0]$ and $g[L_2] = [p^{j_1} e_1, p^{j_2} e_2, p_3^{j_3}]$ with $-\infty < j_1 \leqslant j_2 \leqslant j_3$.

Corollary (Canonical coordinates)

$\exists g \in GL_3 : g[L_1] = [L_0]$ *and* $g[L_2] = \begin{bmatrix} 1 & & \\ & p^l & \\ & & p^j \end{bmatrix}$ *with* $0 \leqslant l \leqslant j$.

Building of $PGL_3(\mathbb{Q}_p)$: apartments and coordinates

Definition (Apartment)

The apartment generated by the basis $\{v_1, v_2, v_3\}$ is the subset of X given by

$$\{[v_1, p^l v_2, p^j v_3],\ l, j \in \mathbb{Z}\}.$$

Fundamental apartment: $\{[e_1, p^l e_2, p^j e_3],\ l, j \in \mathbb{Z}\} = \begin{bmatrix} 1 & & \\ & p^l & \\ & & p^j \end{bmatrix}.$

The previous Corollary on canonical coordinates says that there is $g \in GL_3(\mathcal{F})$ that retracts an apartment onto the fundamental apartment.

Notation

- *Write lattice coordinates* $\langle 1, p^l, p^j \rangle$ *instead of* $[e_1, p^l e_2, p^j e_3]$
- *Canonical coordinates:* $0, l, j$
- *Difference coordinates:* $m = l,\ k = j - l$

Parametrization of the fundamental apartment of $PGL_3(\mathbb{Q}_p)$

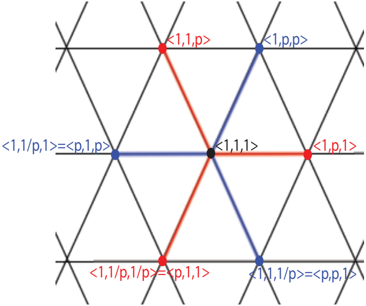


Figure: Lattice coordinates

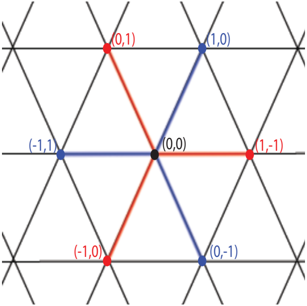


Figure: Difference coordinates

In this figures we see three types of vertices, black, blue and red. Each vertex has neighbors only of the other two types, interleaved.

Buildings for dummies

Those persons in the audience who have been lost with the construction of the affine building $\tilde{A}_2$ can understand such a building euristically as follows.

Think of the integers $\mathbb{Z}$: a 1-dimensional tree. Bifurcate $\mathbb{Z}$ at each integer k: just duplicate the half line at the right (or at the left) of k and fold it sideways, a fixed number of times (say $q-1$). Repeat this procedure at all vertices of $\mathbb{Z}$ and of all the oher vertices obtained by the foldings. At the end you have a homogeneous tree with $q+1$ neighbors of each vertex.

Now start with an apartment in $\tilde{A}_2$, defined just as a hexagonal tiling of $\mathbb{R}^2$ (see the previous Figures). The tiling is created by three bundles of parallel lines, at angles of 120 degrees. Take one of this three bundles and fold the apartment along each of its lines a given number of times (the exact number of times depends on the choice of the prime p in $\mathbb{Q}_p$, but it is the same for all folding lines). Repeat the procedure for each half-plane obtained by the foldings. Then do the same for the other two bundles of parallel lines. What you have at the end is the building $\tilde{A}_2$. Of course the resulting simplicial complex cannot be embedded in $\mathbb{R}^3$.

Buildings for dummies

Those persons in the audience who have been lost with the construction of the affine building $\tilde{A}_2$ can understand such a building euristically as follows.

Think of the integers $\mathbb{Z}$: a 1-dimensional tree. Bifurcate $\mathbb{Z}$ at each integer k: just duplicate the half line at the right (or at the left) of k and fold it sideways, a fixed number of times (say $q-1$). Repeat this procedure at all vertices of $\mathbb{Z}$ and of all the oher vertices obtained by the foldings. At the end you have a homogeneous tree with $q+1$ neighbors of each vertex.

Now start with an apartment in $\tilde{A}_2$, defined just as a hexagonal tiling of $\mathbb{R}^2$ (see the previous Figures). The tiling is created by three bundles of parallel lines, at angles of 120 degrees. Take one of this three bundles and fold the apartment along each of its lines a given number of times (the exact number of times depends on the choice of the prime p in $\mathbb{Q}_p$, but it is the same for all folding lines). Repeat the procedure for each half-plane obtained by the foldings. Then do the same for the other two bundles of parallel lines. What you have at the end is the building $\tilde{A}_2$. Of course the resulting simplicial complex cannot be embedded in $\mathbb{R}^3$.

Building of $PGL_3(\mathbb{Q}_p)$: types

Definition (Type)

Lattice $L = (v_1, v_2, v_3) \leftrightarrow$ matrix $M_L \in GL_3(\mathcal{F})$:

$$M_L = \left[\left(\; v_1 \; \right) \left(\; v_2 \; \right) \left(\; v_3 \; \right) \right]$$

Type $\tau[L]$ of $[L]$ = valuation of $\det(M_L) \mod 3$.

Remark

The type does not depend on the choice of representative in the equivalence class $[L]$*, because* $\det(M_{pL} = p^3 \det(M_L)$*. The determinant is invariant under the action of* GL_3*, hence the type of a vertex* $[L]$ *is invariant under* PGL_3*.*

Weyl chambers

Definition (Fundamental Weyl chamber)

The fundamental Weyl chamber in the fundamental apartment is given by difference coordinates $m, k \geqslant 0$: it is an infinite cone with apex at the origin.

In general, a Weyl chamber is the cone of classes of equivalence of lattices of the type $[L] = [p^{j_1} v_1, p^{j_2} v_2, p^{j_3} v_3]$ with respect to some basis $\{v_1, v_2, v_3\}$ and with $j_1 \leqslant j_2 \leqslant j_3$.

In the building of $PGL_3(\mathbb{Q}_p)$ the fundamental apartment splits into 6 Weyl chambers joining at the origin. They are separated by walls: for instance, the fundamental chamber is the cone of width $\pi/3$ pointing up from $[L_0] = [1, 1, 1]$, and its walls are the half-lines with canonical coordinates $[0, j, j]$ and $[0, 0, j]$ respectively ($j \geqslant 0$).

Remark

The action of PGL_3 is transitive on the Weyl chambers.

Sectors and boundary of the building

Definition (Equivalence of Weyl chambers and boundary)

The intersection of two Weyl chambers is either empty or finite or a wall or a Weyl subchamber (or sector). The last case holds when an infinite subset of the two Weyl chambers belongs to the same apartment, and their walls are parallel.

Two Weyl chambers W_1 and W_2 with a common Weyl subchamber are defined equivalent, $W_1 \sim W_2$. The classes of equivalence are somewhat analogous to ends of graphs and define a boundary Ω of the building X.

Remark

Note that a boundary point is now an infinite cone (sector). It is not an end: each apartment is a lattice (funny: a lattice whose vertices are classes of equivalence of lattices...:-)) and has only one end, but in $\tilde{A}_2$ has six boundary points. Inside a sector one can move to its unique point at infinity along many different paths, also divergent ones: for instance, along the central axis or staying parallel to the walls.

Types and Laplace operators

In a rank 3 affine building (i.e. with apartments of dimension 2) there are several Laplace operators:

Definition (Laplace operators)

- Two Laplacians defined on $F = \{f : V(T) \to \mathbb{C}\}$ and depending on nearest neighbors with a given type shift:

$$L_i f(x) := \frac{1}{J_i} \sum_{y \sim x, \tau(y) = \tau(x) + i \mod 3} f(y)$$

with $i = 1, 2$ and $J_i = |\{y \sim x : \tau(y) = \tau(x) + i \mod 3\}|$;

Definition (continued)

- three Laplacians defined on $F_i = \{f : V_i \to \mathbb{C}\}$ (where $V_i = V_i(T)$ $(i = 0, 1, 2)$ are the vertices of type i) and depending only on nearest neighbors of x in V_i :

$$\triangle_i f(x) = \frac{1}{K_i} \sum_{C_i} f(y)\,,$$

where $x \in V_i$ and $C_i := \{y \in V_i : \exists u \in V_j \quad (j \neq i)$ such that $x \to u \to y$ is a 2-chain (two consecutive edges with $y \neq x)\}$, and $K_i = |C_i|$.

Harmonic and jointly harmonic functions

Remark

- *The operators* L_1 *and* L_2 *commute. On the other end, the operators* $\triangle_i$, $i = 0, 1, 2$, *are defined on different spaces of functions* F_i, *but the canonical injections* $V_i \hookrightarrow V$ *yield injections of* F_i *into* F, *and then the* $\triangle_i$*'s commute.*
- *For the affine building of* PGL_3, *denoted by* $\tilde{A}_2$, *one has* $J_1 = J_2$ *and* $K_0 = K_1 = K_2$, *but there are two other affine buildings of rank 3,* $\tilde{B}_2$ *and* $\tilde{G}_2$, *where this is no longer true. (The other affine buildings are given by the action of suitable* $p-$*adic matrix groups over classes of equivalence of lattices. Higher rank buildings are generated by higher rank semisimple* $p-$*adic groups).*

Definition (Harmonic functions)

- $f : V \to \mathbb{C}$ is jointly harmonic if $L_i f = f$, $I = 1, 2$.
- $f : V \to \mathbb{C}$ is weakly harmonic if $\frac{1}{2}(L_1 + L_2) f = f$.
- $f : V_i \to \mathbb{C}$ is $i-$harmonic if $\triangle_i f = f$.

Apartments in affine buildings of dimension 2 (i.e. rank 3)

Note that the lattice structure in the apartments of the affine buildings of dimension 2 coincides with the corresponding Lie algebra weight structure. In these Figures, the red and blue stars in each apartment are the 1-type-increase and 2-type increase stars of the green vertex v in the center (later denoted by $\Sigma_i^\tau(v)$ for vertices and $\Lambda_i^\tau(v)$ for edges, $i = 1, 2$). The other green dots form its 2-chain star $\Sigma_i(v)$.

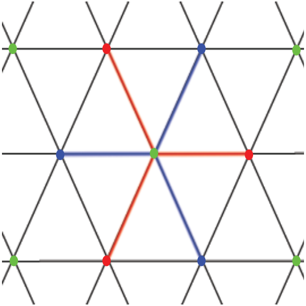


Figure: The apartments of $\tilde{A}_2$

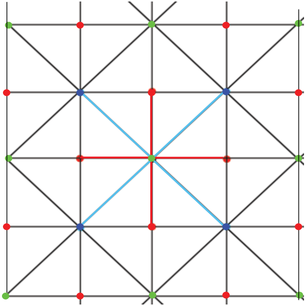


Figure: The apartments of $\tilde{B}_2$

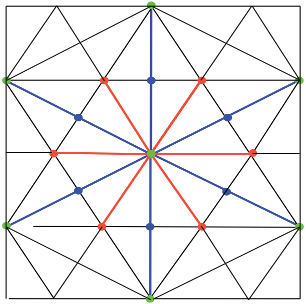


Figure: The apartments of $\tilde{G}_2$

Integral representation of harmonic functions on the boundary of an affine building

Proposition (Cartwright; Mantero–Zappa)

Jointly harmonic functions on an affine building have a Poisson integral representation on the boundary: the Poisson kernel is explicitly known.

Remark

The Poisson integral representation for functions on V_i harmonic with respect to $\triangle_i$ follows by transience of the operator $\triangle_i$. The Poisson kernel has not been computed yet.

Problem

Does the Lusin area theorem hold for jointly harmonic functions on the building of $PGL_3(\mathbb{Q}_p)$? And does it hold for harmonic functions defined on vertices of a given type?

Limitations

Warning

Admissible convergence of harmonic functions on affine buildings of higher rank cannot hold in the exact same way as for trees.

Indeed, choose a boundary point ω, that is a sector. The equivalent of moving at infinity towards ω radially in a tree now becomes moving at infinity along any path within the sector. The simplest jointly harmonic function, the Poisson kernel with pole at ω, diverges to $+\infty$ if one moves out along the central axis, but stays constant if one moves an paths definitively parallel to one of the walls (the constant grows with the distance from the wall). Therefore there is no radial limit if we allow all such paths.

So, radial convergence (and a fortiory admissible convergence) should be intended in the sense that we limit attention to paths that wander off the central axis of a sector only within bounded distance.

Harmonicity of index stabilizers

There are two ingredients that we need in order to transport to buildings the argument that we developed on trees:

- Call $K \subset PGL_3(\mathbb{Q}_p)$ the stability subgroup of $[L_0]$, that is, $K = PGL_3(\mathcal{O})$, and $w = [L_{mn}]$ the class of lattices with difference coordinates m, n, and $K_w = K_{mn} \subset K$ the stability subgroup in K of $[L_{mn}]$: then the index $[K : K_w]$ should be a jointly harmonic, or $i-$harmonic function of w.
- A suitable Green formula should hold.

Proposition (L. Atanasi)

The map $w \mapsto [K : K_w]$ is jointly harmonic.

Forthcoming task, still to be done

It has not yet been computed if the stability subgroup index map is $i-$harmonic (that is, if it is in the kernel of $\triangle_i$ when restricted to V_i).

Terminology recap: 2-chain star of a vertex

Remind: two chains are equivalent if they have the same beginning vertices and the same end vertices.

Definition (2-chain star of a vertex)

Given a vertex (i.e., class of equivalence of lattices) $v = [L] \in V_i$, the V_i-neighbors of v are the vertices connected to v by a 2-chain. The set of $i-$neighbors of v is the $i-$star $\Sigma_i(v)$ of v. We write $u \sim_i v$ if u and v are $i-$neighbors. Observe that the neighbors are in 1-1 correspondence with the equivalence classes $\langle\sigma\rangle$ of 2-chains σ. This correspondence maps $\Sigma_i(v)$ to the 2-chain-star $\Lambda_i(v)$. Remember that the cardinalities of these sets are K_i. Θ_i denotes the set of all 2-chains with both endpoints in V_i.

Terminology recap: type-shift star of a vertex

Definition (type-increasing star of a vertex)

The $i-$type-shift vertex star $\Sigma_i^{\tau}(v)$ of a vertex v is the set of vertices $u \sim v$ such that $\tau(u) = \tau(v) + i \mod 3$. The edges from v to these vertices form the $i-$type-shift edge star $\Lambda_i^{\tau}(v)$. We recall that $|\Sigma_i^{\tau}(v)| = |\Lambda_i^{\tau}(v)| = J_{\tau(v)+i}$. But in the building $\tilde{A}_2$ of PGL_2 all types have the same valency, say H, and so $|\Sigma_i^{\tau}(v)| = |\Lambda_i^{\tau}(v)| = H$: we shall limit attention to this case.

Gradients

Definition

- *2-chain gradient*: For any function $f : V_i \to \mathbb{R}$, the gradient $\nabla_i f : \Theta_i \to \mathbb{R}$ is

$$\nabla_i f(\sigma) = f(e(\sigma)) - f(b(\sigma)).$$

- *type-shift gradient*: For any $f : V(T) \to \mathbb{R}$, $v \in V(T)$ and $\sigma \in \Lambda_i^\tau(v)$, the gradient $\nabla_i^\tau f(v) : \Lambda \to \mathbb{R}$ is

$$\nabla_i^\tau f(\sigma) = f(e(\sigma)) - f(b(\sigma)) = f(e(\sigma)) - f(v).$$

Definition

For $v \in V_i$, let

$$\|\nabla_i f(v)\|^2 = \frac{1}{K_i} \sum_{\langle\sigma\rangle \in \Lambda_i(v)} |\nabla_i f(\sigma)|^2 = \frac{1}{K_i} \sum_{u \sim_i v} |\nabla_i f(\sigma)|^2 \,,$$

and define similarly $\|\nabla_i^\tau f(v)\|^2$, with $\Lambda_i^\tau(v)$ in place of $\Lambda_i(v)$.

Corollary

- *If $v \in V_i$ then*

$$\triangle_i f(v) = \left(\frac{1}{K_i} \sum_{u \sim_i v} f(u) \right) - f(v) = \frac{1}{K_i} \sum_{\langle\sigma\rangle \in \Lambda_i(v)} \nabla_i f(\sigma) \,.$$

- *For $v \in V$, setting again $\Lambda_i^\tau(v) = \{u \sim v : \tau(u) = \tau(v) + i\}$,*

$$L_i f(v) = \left(\frac{1}{J_{\tau(v)+i}} \sum_{u \in \Lambda_i^\tau(v)} f(u) \right) - f(v) \,.$$

As in the continuous case one finds

$$\triangle_i(f^2) = \|\nabla_i f\|^2 + 2f\triangle_i f$$
$$L_i(f^2) = \|\nabla_i^\tau f\|^2 + 2fL_i f$$

and of course the right hand sides are $\|\nabla_i f\|^2$ (respectively, $\|\nabla_i^\tau f\|^2$) if f is $i-$harmonic (respectively, L_i-harmonic or jointly harmonic).

Boundary of a finite set

Definition

- Boundary of a finite subset of $V(T)$: the *boundary* ∂Q of a subset $Q \subset V$ is $\partial Q = \{\sigma \in \Lambda, b(\sigma) \in Q, e(\sigma) \notin Q\}$.
- Boundary of a finite subset of V_i: the *boundary* ∂Q of a subset $Q \subset V_i$ is $\partial Q = \{\langle \sigma \rangle : \sigma \in \Theta_i, b(\sigma) \in Q, e(\sigma) \notin Q\}$.

The *trace* $b(A)$ of a subset $A \subset \Lambda$ is $b(A) = \{b(\sigma) : \sigma \in A\}$. For $Q \subset V$, the trace of ∂Q is also called the *frontier* $\mathfrak{F}Q$.

It is now clear how to define the area functions in both cases (for functions defined on a single type, or else defined everywhere but for type-shift Laplacians). We omit the definition.

Warning

At each boundary point ω, the area functions are defined as sums of the norm square of the gradient, but remember, we should only sum over sets of vertices at bounded distance from the central axis of the sector ω.

Green formulas on buildings

One should now look for two possible extensions of the Green formula to buildings. Is the following true?

(*i*) Let T be the building $\tilde{A}_2$ and f and h functions on $V(T)$. If Q is a finite subset of T, then

$$\sum_Q (hL_i f - fL_i h) = \frac{1}{H} \sum_{v\in\mathfrak{F}Q,\, \sigma\in\Lambda_i^\tau(v)\cap\partial Q} \left(h\circ b\nabla_i^\tau f - f\circ b\nabla_i^\tau h\right). \text{[1]}$$

(*ii*) If f and h are functions on V_i and Q is a finite subset of V_i, $i=1,2$, then

$$\sum_Q (h\triangle_i f - f\triangle_i h) = \frac{1}{K_i} \sum_{\partial Q} (h\circ b\nabla_i f - f\circ b\nabla_i h).$$

[1] Here we collect a factor H because valency is constant in $\tilde{A}_3$. On the other affine buildings of rank 3 this is not so, and the proposed formula becomes much more complicated.

Proposition

In the previous question, (i) is false but (ii) is true.

Sketch of proof. First deal with the case where Q is a singleton (easy). Then by induction: we suppose it true for some finite set Q and add a singleton $\{v\}$. If v is far from Q then both sides are additive, but if v is at distance 1 from Q then only the left side is clearly additive: the second is not because the boundary of the union is a proper subset of the union of the boundary: the edges from Q to v are not any longer in the boundary of the union, and neither are those from Q to v (oriented in the opposite direction). But the right side in (ii) is antisymmetric with respect to edge orientation (like a differential form!), and so there are cancelations and the Green formula holds. Instead the right side in (i) does not yield cancelations, because reversing the edge orientation changes the cyclic order of the types, so it mixes the types. The Green formula would only hold for the sum of the separate Laplacians L_i, and *it does lead to a Lusin area function for weakly harmonic functions only*. □

Conclusion

The proposed approach to the Lusin area theorem fails for type-shift Laplacians because the Green formula fails. It should work for $i-$harmonic functions if the index function of the stability subgroups is $i-$harmonic.

This is a satisfactory result but not as challenging, since a function that is $i-$harmonic for all types i is just a collection of 3 independent, uncorrelated functions. This result is just a bit more difficult (but technically much more complicated) than a Lusin area function for weakly harmonic functions only (that is, with respect to the untyped operator $L = \frac{1}{2}(L_1 + L_2)$). It works with the same ingredients of the proof for a homogeneous tree (or a rank one symmetric space, the half-plane for instance).

What is left

Instead, the proof of a Lusin theorem for jointly harmonic functions does not follow from the same ingredients because the Green formula cannot be written in the same way.

Problem

Can one write a suitable modification of the Green formula and the area function that make the theorem true for jointly harmonic functions?

If so, this would give some ideas on how to tackle the same question and to define the area function on higher rank symmetric spaces (Malliavin & Malliavin (1977) for products of the half-plane, and Korányi & Putz (1981) for products of rank one symmetric spaces).

Indeed, this is the interest and motivation for the problems introduced and partially answered in this presentation, that stem out of a question raised by Adam Korányi.

References

L. Atanasi, M. A. Picardello *The Lusin area function and local admissible convergence of harmonic functions on homogeneous trees*, Trans. Amer. Math. Soc. **360** (2008), 3327–3343.

P. Cartier, *Fonctions harmoniques sur un arbre*, Symp. Math. **9** (1972), 203–270.

F. Di Biase, M. A. Picardello, *The Green formula and H^p Spaces on trees*, Math. Z. **218** (1995), 253–272.

A. Korányi, M. A. Picardello, *Boundary behaviour of eigenfunctions of the Laplace operator on trees*, Annali Scuola Normale Superiore - Pisa (Serie IV), **13** (1986), 3, 389–399.

A. Korányi, M. A. Picardello, M. H. Taibleson, *Hardy spaces on non-homogeneous trees*, Symp. Math. **29** (1988), 206–265.

- A. Korányi, R. B. Putz, *Local Fatou theorem and area theorem for symmetric spaces of rank one*, Trans. Amer. Math. Soc. **224** (1976), 1, 157-168.
- A. Koràny, R. B. Putz, *An area theorem for products of symmetric spaces of rank one*, Bull. Sc. Math., 2e serie, **105** (1981), 3-16.
- M. A. Picardello, *Local admissible convergence of harmonic functions on non-homogeneous trees*, submitted for publication.
- M. P. Malliavin, P. Malliavin, *Intégrales de Lusin–Calderón pour les fonctions biharmoniques*, Bull. Sc. Math., 2e serie, **101** (1977), 357-384.
- A.M. Mantero, A. Zappa, *papers on buildings, since 1997*.
- A.M. Mantero, T. Steger, A. Zappa, in preparation.
- J.P. Serre, *Arbres, amalgames, SL_2*, Asterisque **46**, 1977.
- T. Steger, *Local Fields and Buildings*, Contemporary Math. **206** (1997), 79–107 *(redaction by S. Sawyer)*.

- D. Cartwright, *A brief introduction to buildings*, Contemporary Math. **206** (1997).
- D. Cartwright, *Harmonic functions on buildings of type A_n*, in *Random Walks and Discrete Potential Theory*, M.Picardello, W.Woess (eds.), Symp. Math. **39**, 104–138.
- K.S. Brown, *Buildings*, Springer, 1989.
- M. Ronan, *Lectures on Buildings*, Academic Press, 1989.
- F. Bruhat, J. Tits, *Groupes réductifs sur un corps local, I. Données radicielles valuées*, Publ. Math. IHES **41** (1972), 5–251.