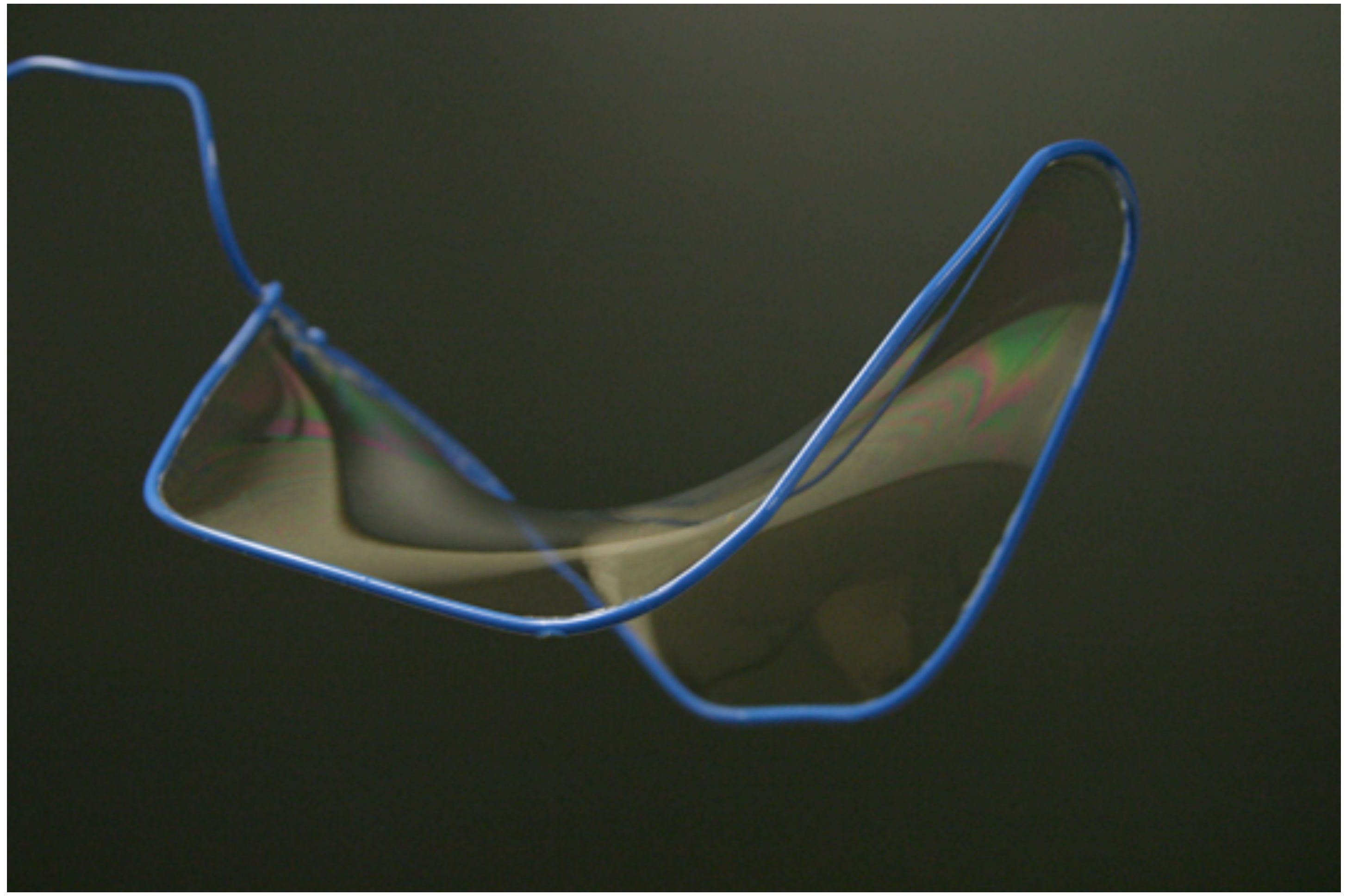
University of BRISTOL

How wild can a group with a quadratic Dehn function be?

Tim Riley

"Boundaries"
Graz

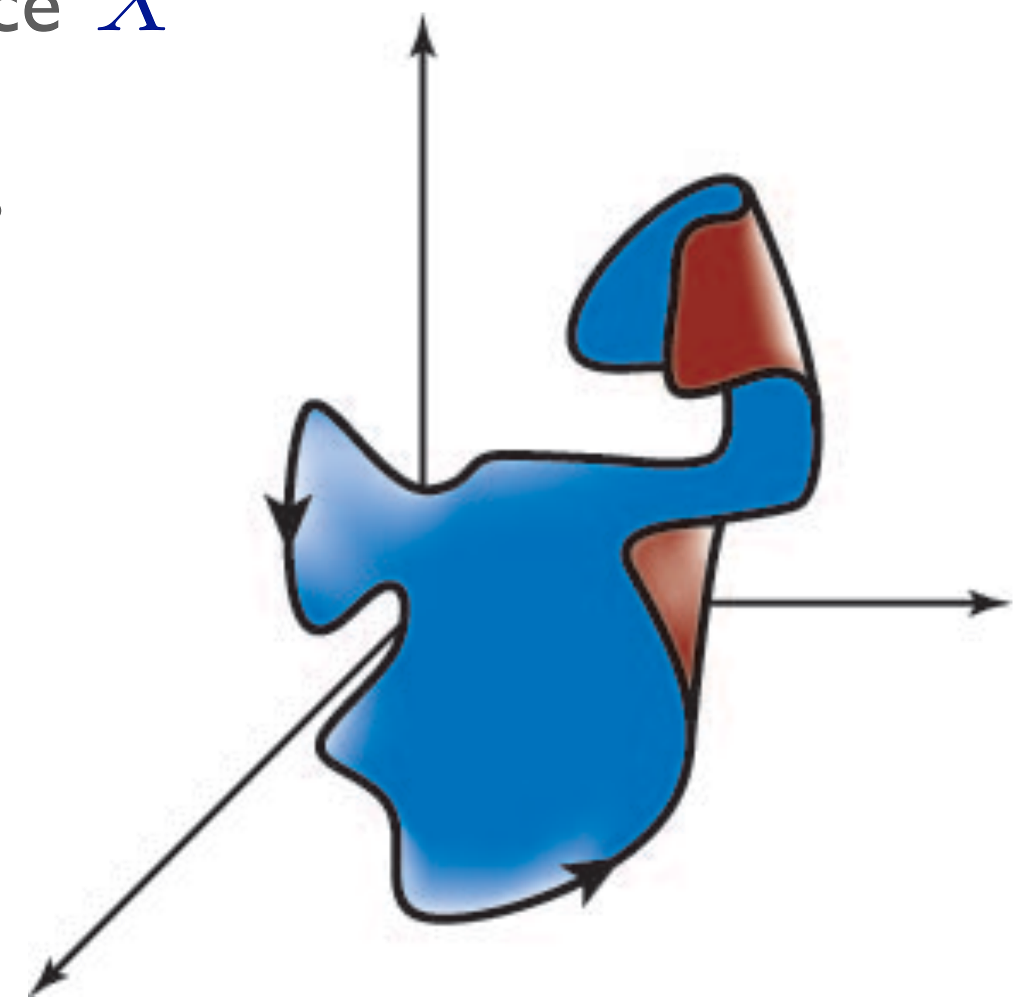
June 30, 2009



Euclidean space "enjoys a quadratic isoperimetric function."

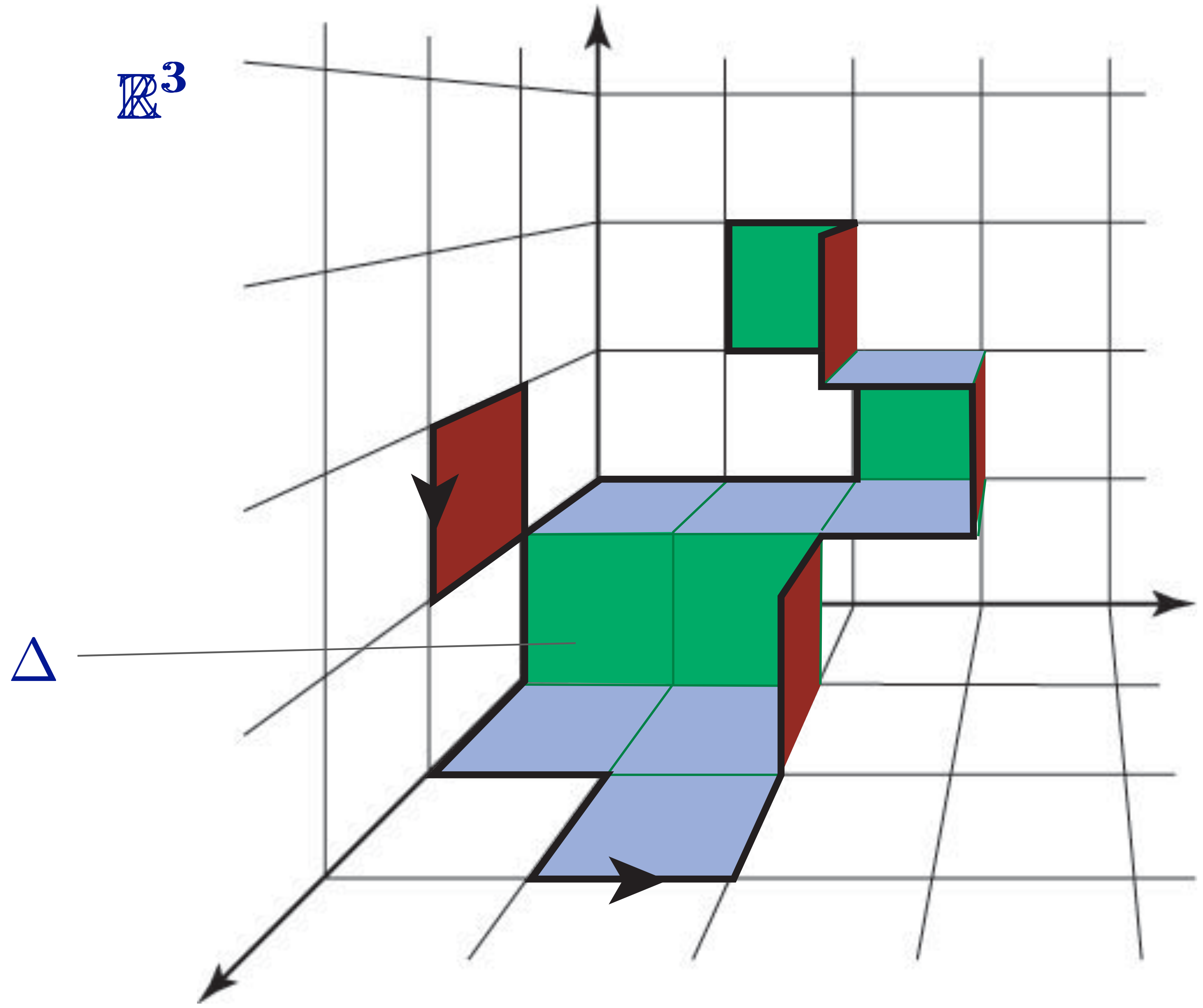
ρ a loop in a simply connected space X

$\mathrm{Area}(\rho)$ is the infimum of the areas of discs spanning ρ.



$\mathrm{Area}_X : [0, \infty) \to [0, \infty]$ is defined by

$$\mathrm{Area}_X(l) \ = \ \sup\{\mathrm{Area}(\rho) \mid \ell(\rho) \leq l\}.$$

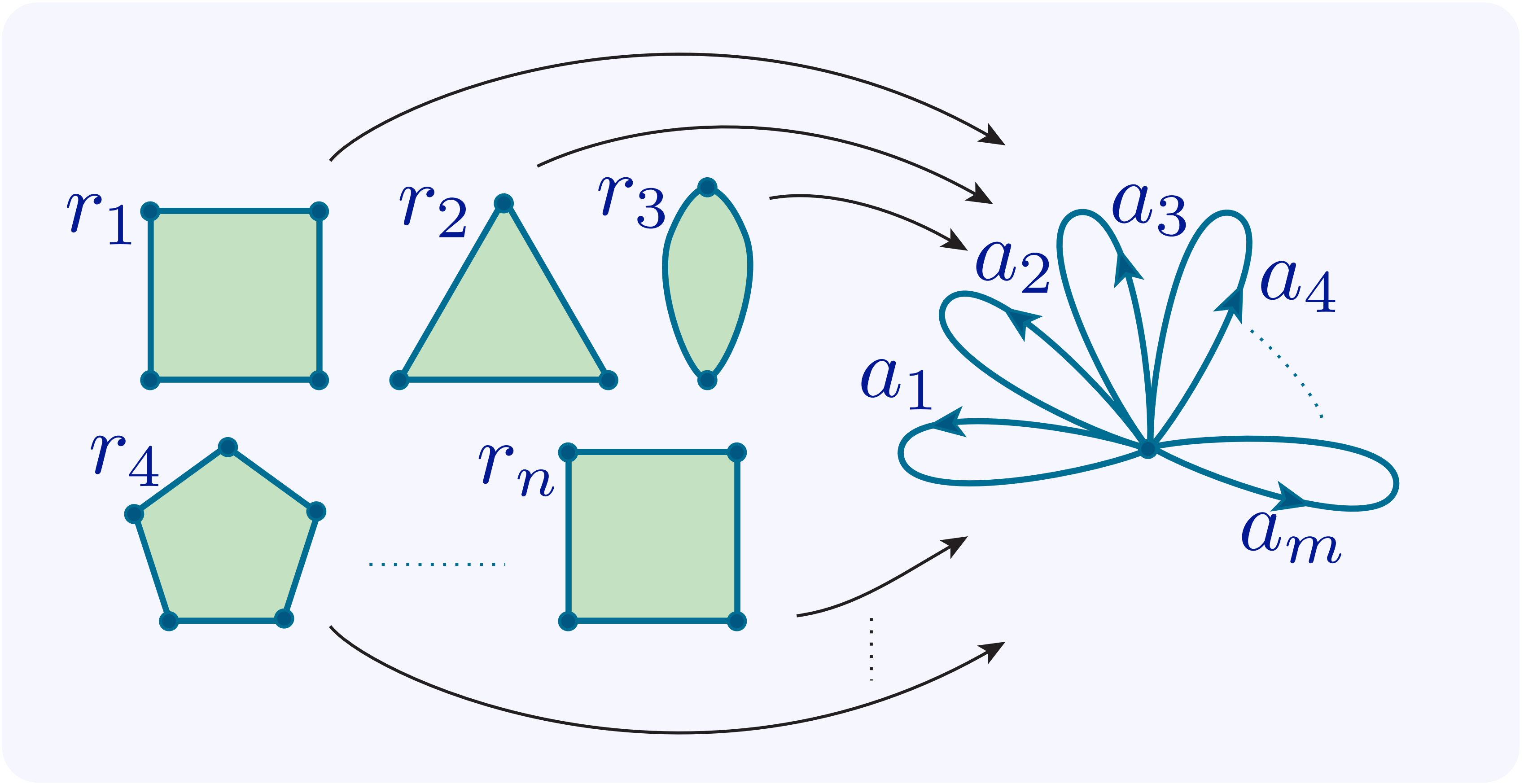


$\mathrm{Area}(\Delta) = \#$ 2-cells

$\mathcal{P} = \langle a_1, \ldots, a_m \mid r_1, \ldots, r_n \rangle$ a finite presentation of a group Γ

The presentation 2-complex of $\mathcal{P}$:

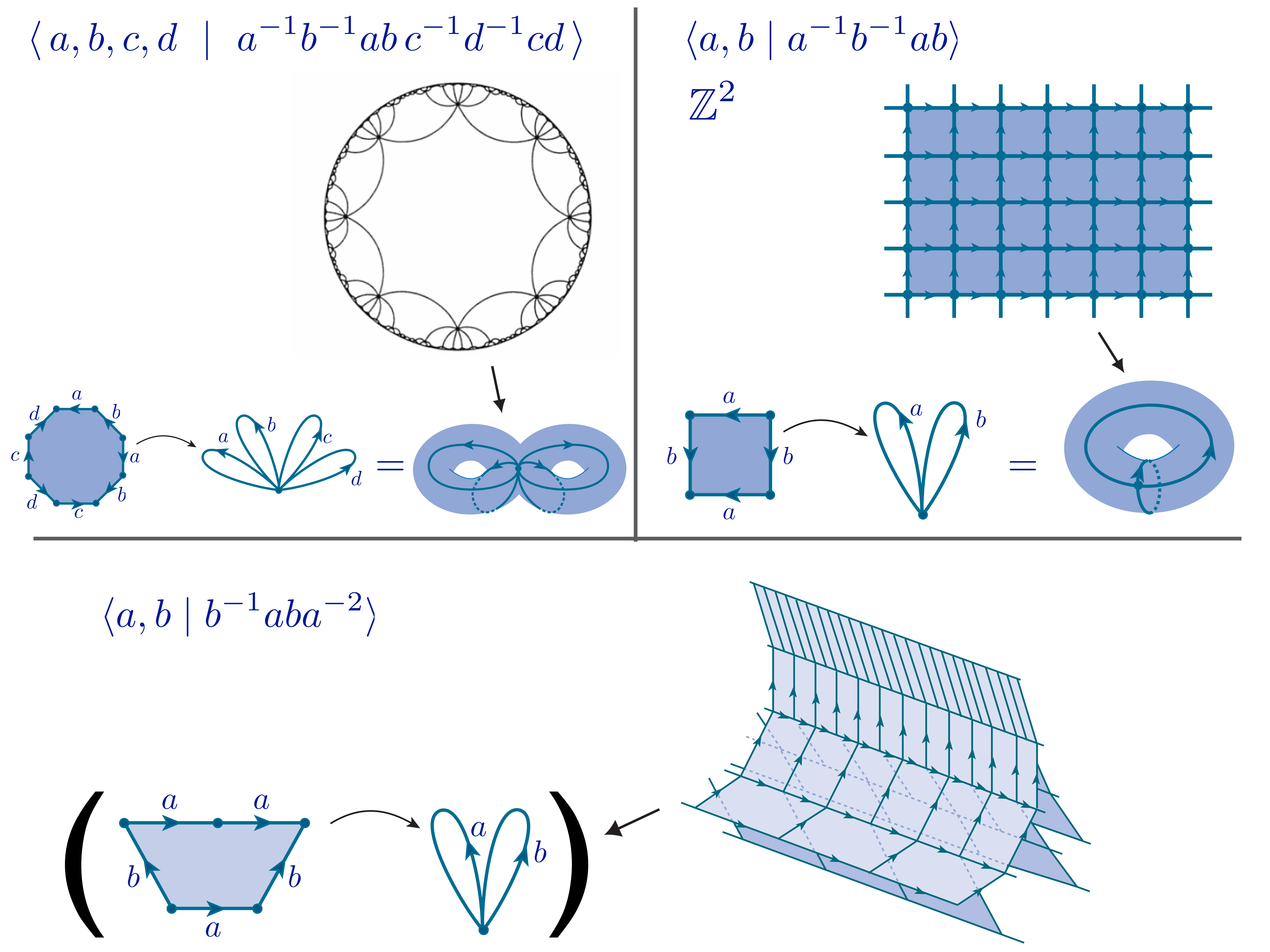
$K =$



$\pi_1(K) = \Gamma$

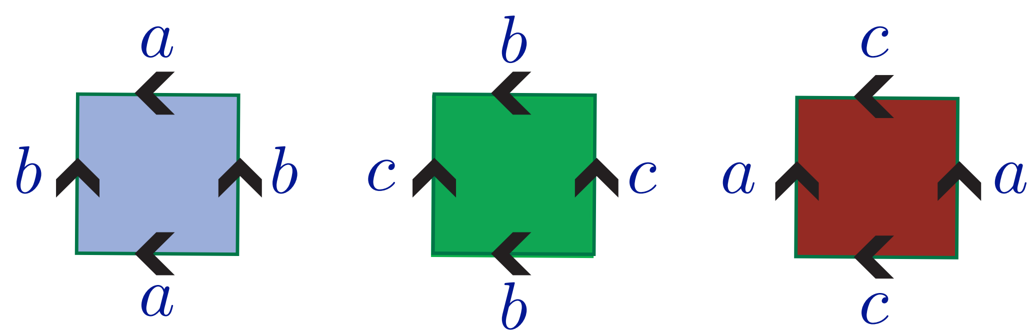
The universal cover $\widetilde{K}$ is the *Cayley 2-complex* of $\mathcal{P}$.

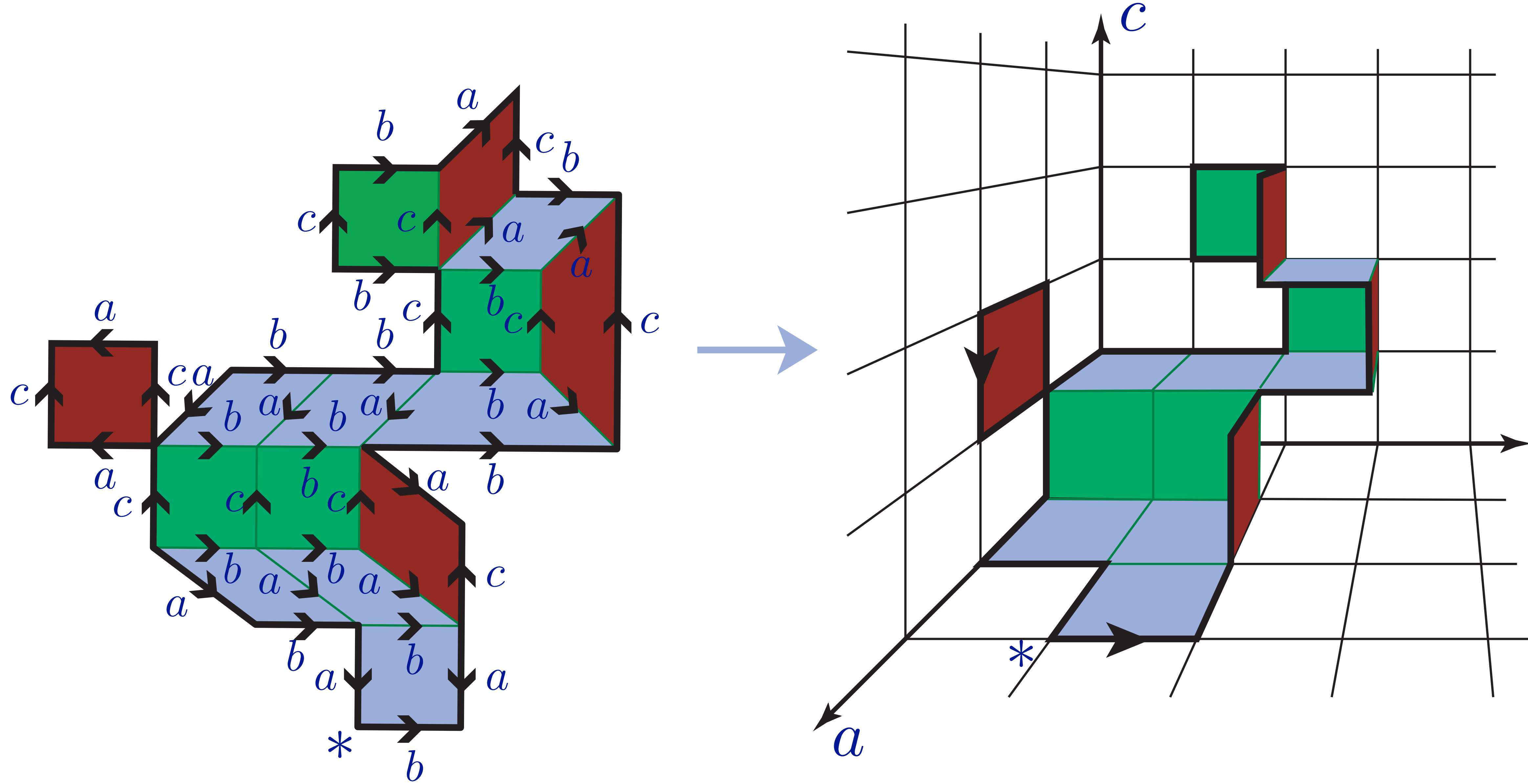
Its 1-skeleton $\widetilde{K}^{(1)}$ is the *Cayley graph* of $\mathcal{P}$.



$\langle a, b, c, d \mid a^{-1}b^{-1}ab\,c^{-1}d^{-1}cd \rangle$
$\langle a, b \mid a^{-1}b^{-1}ab \rangle$
$\mathbb{Z}^2$
$\langle a, b \mid b^{-1}aba^{-2} \rangle$
a
b
c
d
=

$\mathbb{Z}^3$ $\langle a, b, c \mid [a, b], [b, c], [c, a] \rangle$





A *van Kampen diagram*

For an edge-loop ρ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\mathrm{Area}(\rho)$ is the minimum of $\mathrm{Area}(\Delta)$ over all van Kampen diagrams spanning ρ.

The *Dehn function* $\mathrm{Area}_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\widetilde{K}$ is

$$\mathrm{Area}_{\mathcal{P}}(n) \;=\; \max\{\mathrm{Area}(\rho) \mid \text{edge-loops } \rho \text{ in } \widetilde{K} \text{ with } \ell(\rho) \leq n\}.$$

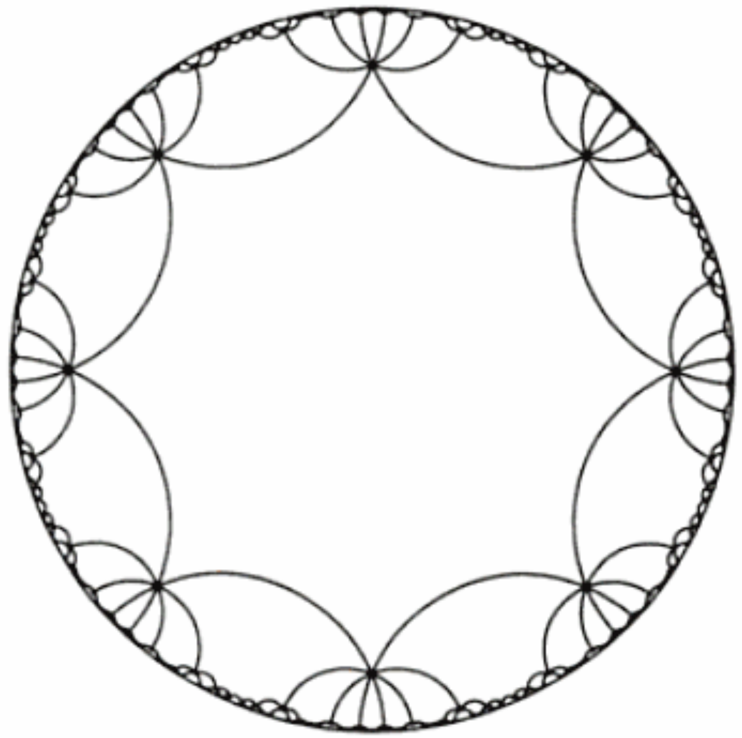
The Filling Theorem. If $\mathcal{P}$ is a finite presentation of the fundamental group of a closed Riemannian manifold M then

$$\mathrm{Area}_{\mathcal{P}} \;\simeq\; \mathrm{Area}_{\widetilde{M}}.$$

$f \preceq g$ when $\exists C > 0$ such that $\forall n > 0,\ f(n) \leq Cg(Cn + C) + Cn + C$.

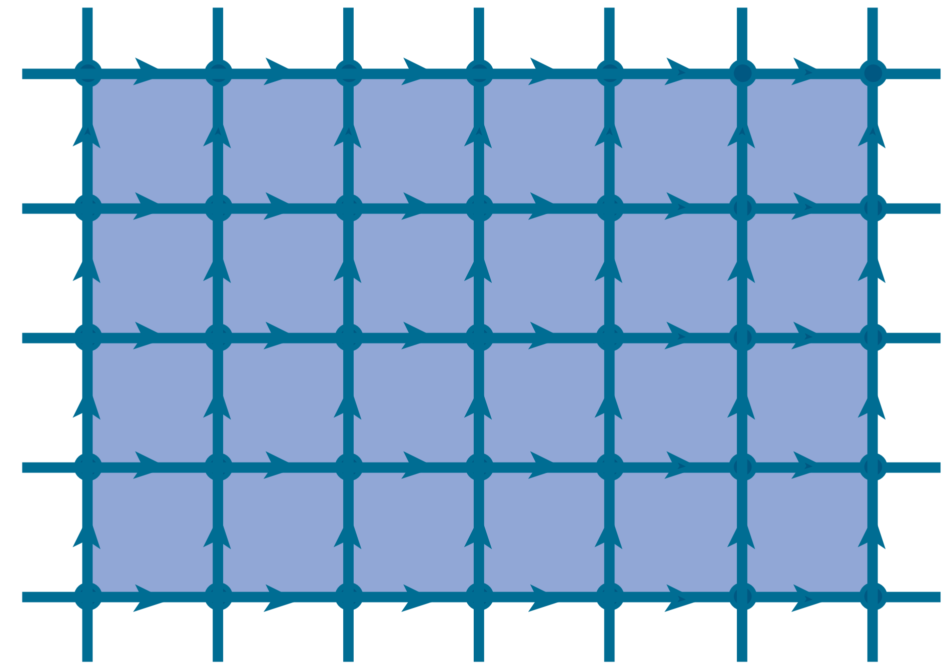
$f \simeq g$ when $f \preceq g$ and $g \simeq f$.

$\langle a, b, c, d \mid a^{-1}b^{-1}ab\,c^{-1}d^{-1}cd \rangle$



$\mathrm{Area}(n) \simeq n$

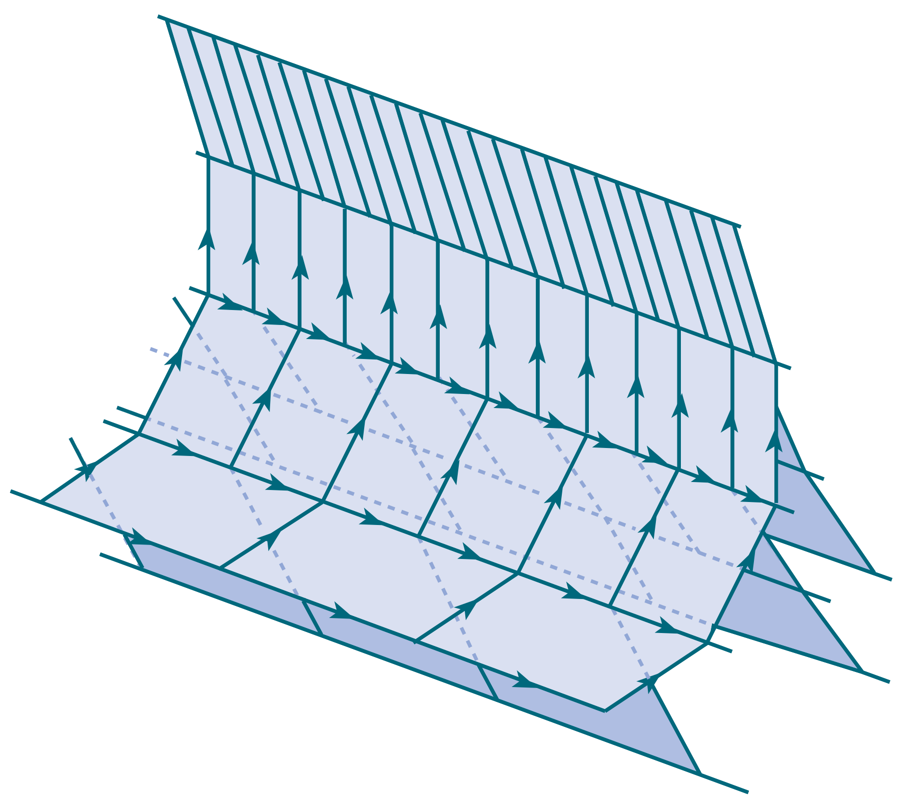
$\langle a, b \mid a^{-1}b^{-1}ab \rangle$

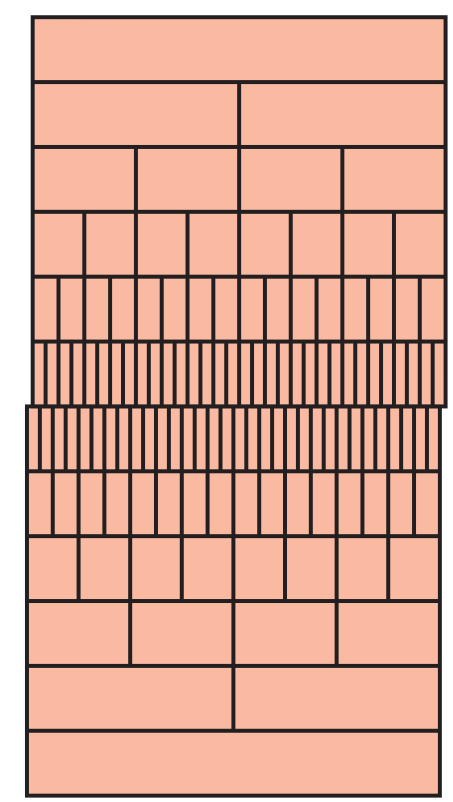


$\mathrm{Area}(n) \simeq n^2$

$\langle a, b \mid b^{-1}aba^{-2} \rangle$

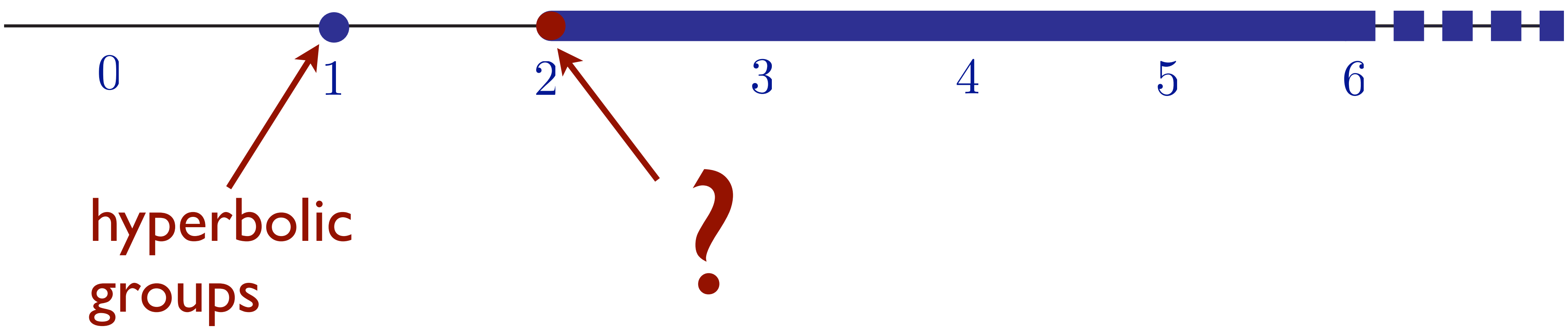
$\mathrm{Area}(n) \simeq 2^n$





$$\mathrm{IP} \; = \; \{ \, \alpha > 0 \mid n \mapsto n^{\alpha} \text{ is } \simeq \text{ a Dehn function } \}$$

The closure of IP is



Gromov, Bowditch, N. Brady, Bridson, Olshanskii, Sapir...

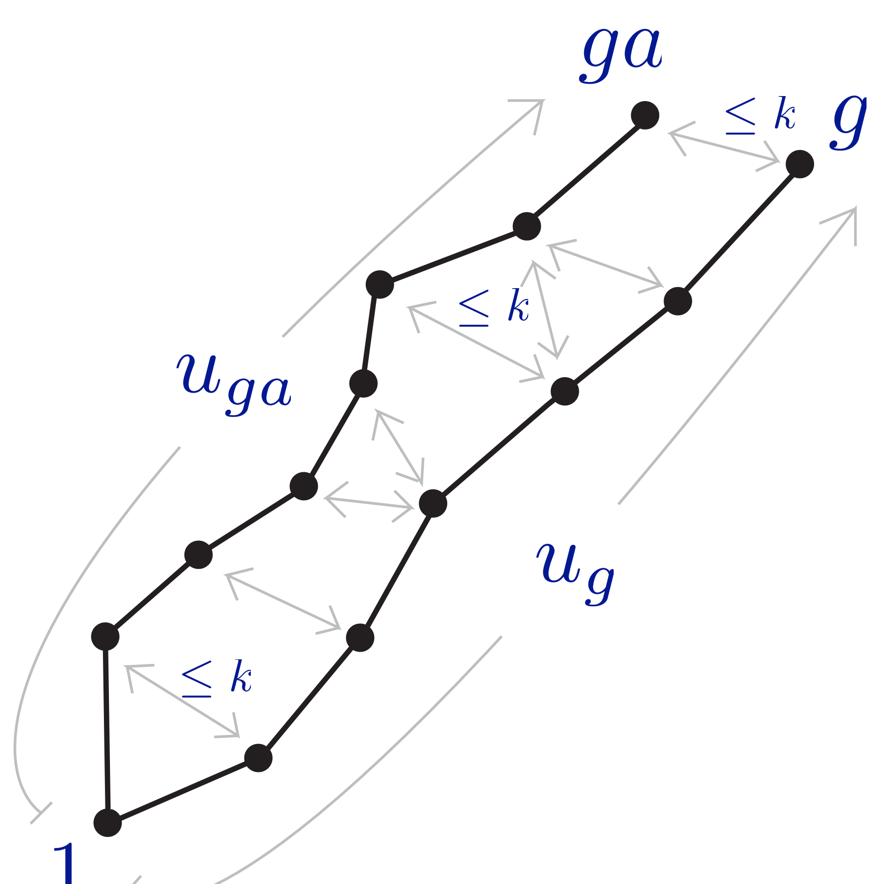
"Non–positively curved" groups

Γ a group with finite generating set $\mathcal{A}$

An *asynchronously* k*–fellow–travelling linear–length combing* is a choice of words u_g for each $g \in \Gamma$, such that $u_g = g$,

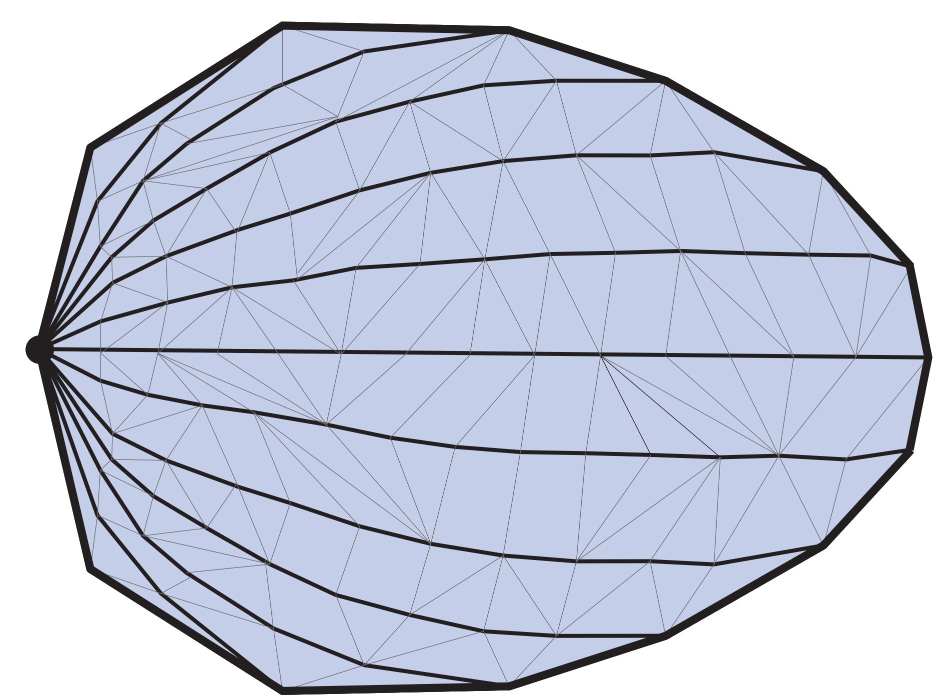
$$\ell(u_g) \leq constant \cdot d(1, g)$$

and $\forall g \in \Gamma$, $\forall a \in \mathcal{A}^{\pm 1}$,



e.g.
- CAT(0) groups
- semihyperbolic groups
- automatic groups

Coning yields a quadratic isoperimetric function:



$$\text{Area}(n) \preceq n^2$$

Free–by–cyclic groups

$F_n \rtimes_\phi \mathbb{Z}$

e.g. $F_3 = F(a,b,c)$ with $\phi : \begin{cases} a \mapsto a \\ b \mapsto ab \\ c \mapsto a^2c \end{cases}$ is neither $\mathrm{CAT}(0)$ nor automatic.

[Brady, Bridson, Gersten, Reeves]

Theorem. [Bridson–Groves] Free–by–cyclic groups enjoy quadratic isoperimetric functions.

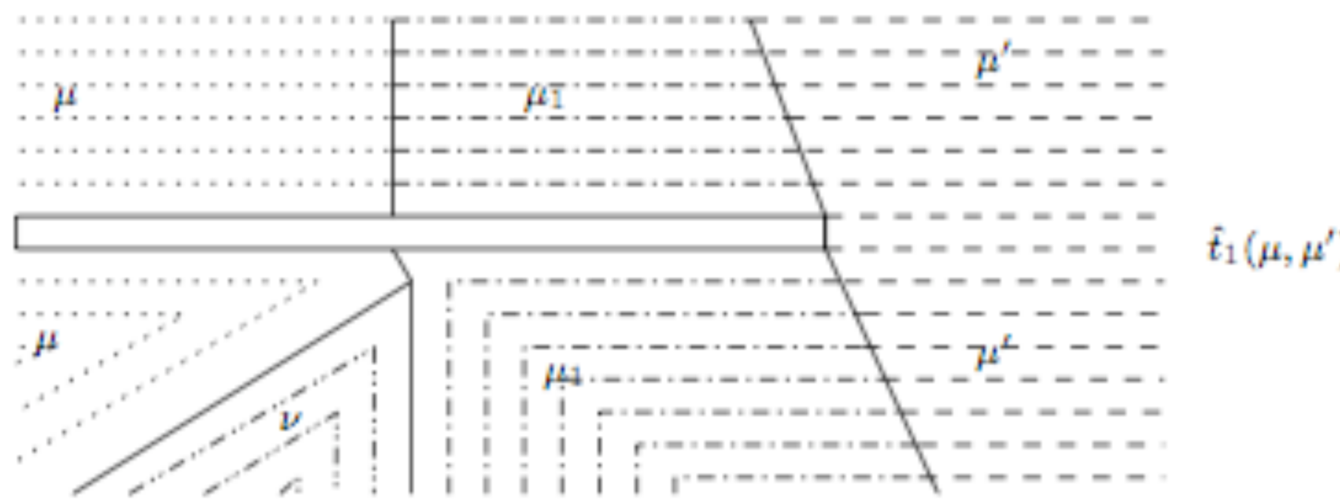


FIGURE 19. A team of genesis (G2)

To appear as a monograph in the *Memoirs of the AMS* series. 188 pages!!!

Nilpotent Groups

Example. The class c free nilpotent group on two generators has Dehn function $\simeq n^{c+1}$.

[Baumslag–Miller–Short, Gersten, Gromov, Pittet]

Example. The 5–dimensional (class 2) integral Heisenberg group

$$\begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & 0 & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has Dehn function $\simeq n^2$.

[Thurston, Gromov, Allcock, Olshanskii–Sapir]

Example. The m–jet bundle, $J^m(\mathbb{R}^k)$ is a class–$(m+1)$ nilpotent group and enjoys Euclidean isoperimetric functions for fillings of i–cycles whenever $i \leq k$.

[Young]

Solvable Groups

[Drutu; Leuzinger–Pittet] Certain semidirect products $N \rtimes A$ of a nilpotent and an abelian simply connected Lie group admit quadratic isoperimetric functions.* These include, for $n \geq 3$,

$$\mathrm{Sol}_{2n-1} = \mathbb{R}^n \rtimes \mathbb{R}^{n-1} = \left\{ \begin{pmatrix} e^{t_1} & 0 & \cdots & 0 & x_1 \\ 0 & e^{t_2} & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & e^{t_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} x_i, t_i \in \mathbb{R}, \\ \\ \sum_{i=1}^{n} t_i = 0 \end{array} \right\}$$

[cf. Gromov]

and

$$\left\{ \begin{pmatrix} e^{t_1} & x_1 & x_4 & x_6 \\ 0 & e^{t_2} & x_2 & x_5 \\ 0 & 0 & e^{t_3} & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \begin{array}{l} x_i, t_i \in \mathbb{R}, \\ \\ t_1 + t_2 + t_3 = 0 \end{array} \right\},$$

which is isometric to a horosphere in $\mathrm{SL}_4(\mathbb{R})/\mathrm{SO}_4(\mathbb{R})$.

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* In fact, Drutu's result is more general.

Discrete examples:

Corollary. There are polycyclic semi-direct products $\mathbb{Z}^n \rtimes \mathbb{Z}^{n-1}$ (cocompact lattices in Sol_{2n-1}) with exponential growth (so not virtually nilpotent) and quadratic Dehn functions.

[Leuzinger–Pittet]

Stallings' Group

John R. Stallings

A FINITELY PRESENTED GROUP WHOSE 3-DIMENSIONAL INTEGRAL HOMOLOGY IS NOT FINITELY GENERATED.*

By John Stallings.[1]

The aim of this note is to provide a counterexample to a conjecture about the niceness of finitely presented groups. This also gives a counterexample to a conjecture about $\pi_2(K)$, where K is a finite complex.

Example. *The group G with presentation*

$$\{a, b, c, x, y : [x, a], [y, a], [x, b], [y, b], [a^{-1}x, c], [a^{-1}y, c], [b^{-1}a, c]\}$$

with five generators and seven relations has as its 3-dimensional homology group with integer coefficients a not finitely generated group. (Note: $[u, v] = uvu^{-1}v^{-1}$.)

Corollary 1. *There is no projective resolution* [1] *of Z over $Z(G)$ which is finitely generated in dimension 3.*

Corollary 2. *If K is any finite complex with $\pi_1(K) \approx G$, then $\pi_2(K)$ is not finitely generated, even as a module over $\pi_1(K)$.*

Amer. J. Math., Vol. 85, No. 4, 1963, page 541

Bieri–Stallings groups:

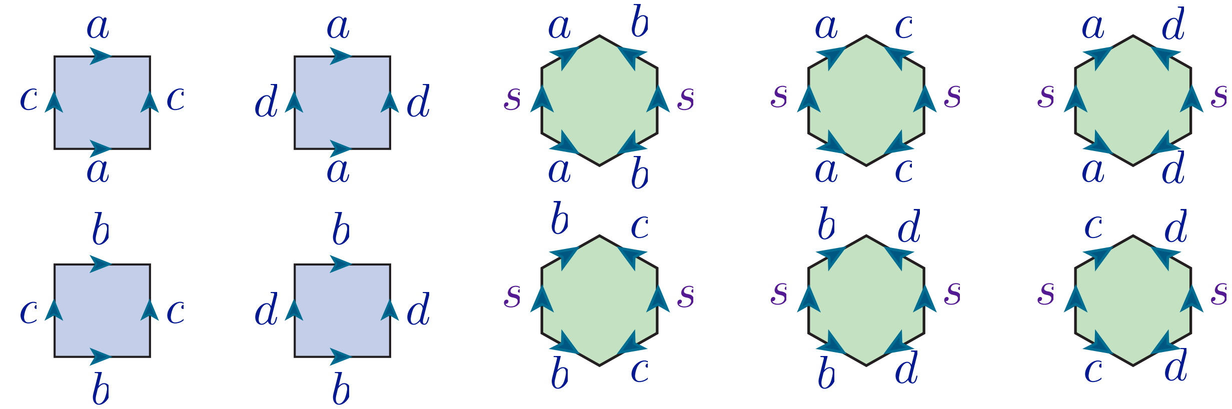
$$\begin{aligned} \mathrm{Ker}\,(F(a_1,b_1) \times \cdots \times F(a_n,b_n) \;&\to\; \mathbb{Z}) \\ a_i, b_i \;&\mapsto\; 1,\ \forall i \end{aligned}$$

are of *Type* F_{n-1} but not *Type* F_n.

Stallings' group is the case $n = 3$.

We will work with a presentation of Stallings' group as an HNN–extension of $F(a,b) \times F(c,d)$:

$$S \;=\; \left\langle \; a, b, c, d, s \;\middle|\; \right.$$



$$\left. \right\rangle$$

The Dehn function of Stallings' group is...

...at most polynomial [Gersten, 1995]

... $\preceq n^5$ [Gersten]

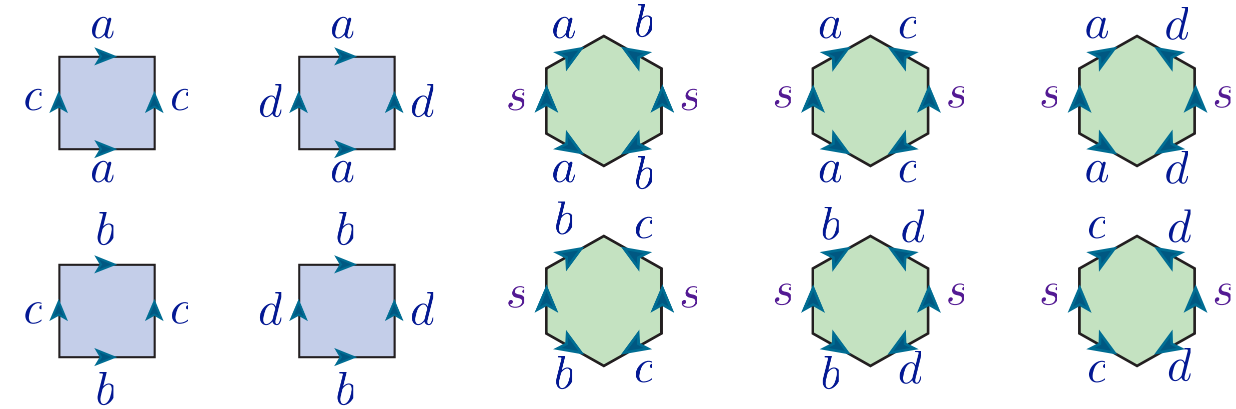
... $\preceq n^3$ [Baumslag–Bridson–Miller–Short, 1997]

... $\preceq n^{5/2}$ [Elder–R.]

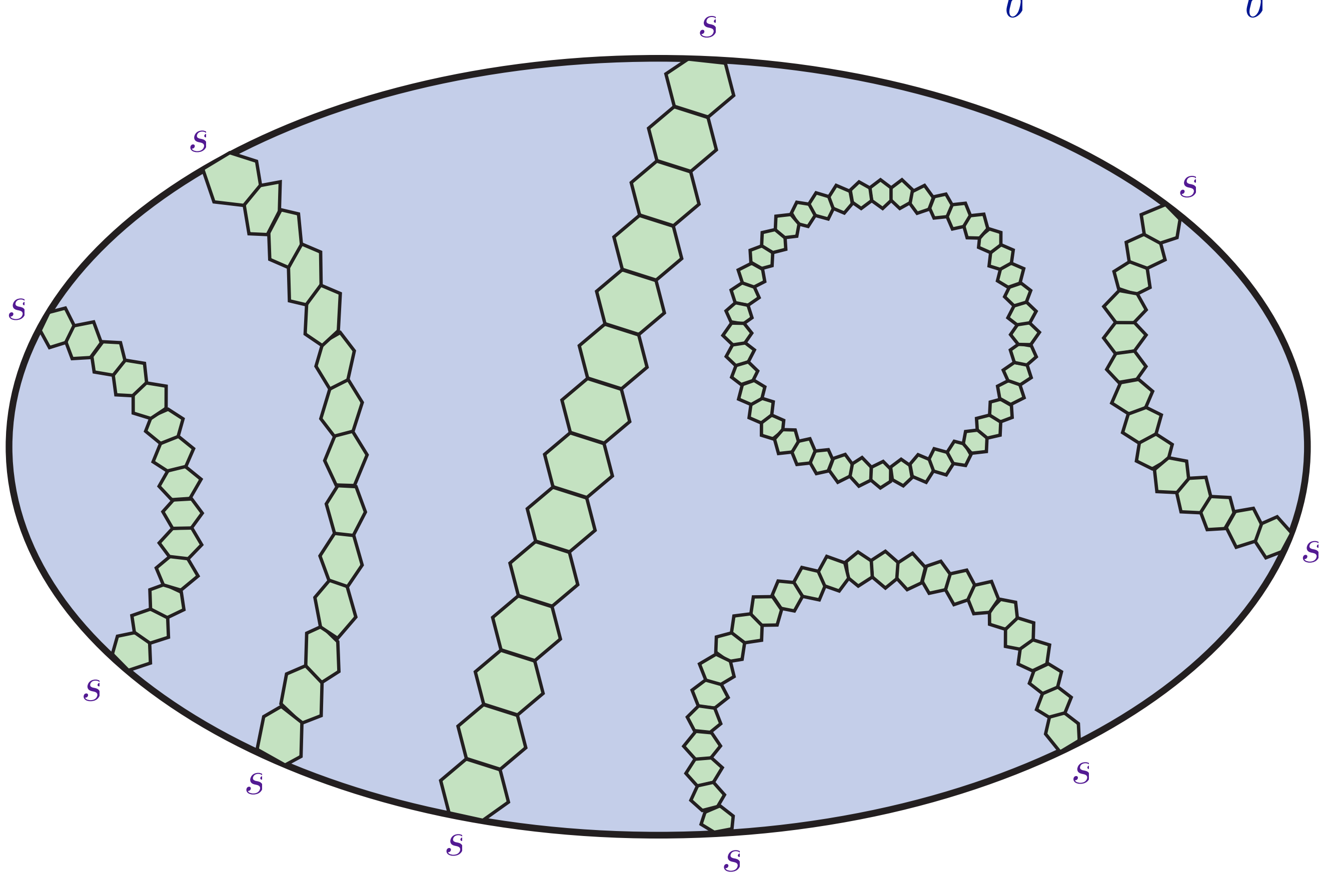
... $\preceq n^{7/3}$ [Dison–Elder–R.]

... quadratic [Dison–Elder–R.–Young].

$\Big\langle\ a, b, c, d, s\ \Big|$



$\Big\rangle$



The words along the sides of the s–corridors are *alternating*.

Put words with zero exponent–sum into alternating form:

$$u \ = \ a^{-1}c^{-1}ab\,d^{-1}a\,c^{-1}d$$

$$\to a^{-1}ab\,a\,c^{-1}d^{-1}c^{-1}d$$

$$\to ca^{-1}ac^{-1}\,bc^{-1}\,ac^{-1} \ \ (ac^{-1})^{-2} \ \ ac^{-1}ad^{-1}ac^{-1}da^{-1} \ = \ \hat{u}$$

Cost is $\preceq \ell(u)^2$ and $\ell(\hat{u}) \ \leq \ 4\ell(u)$. Next, divide and conquer very carefully!

Thompson's Group F

$$F \;=\; \langle\, a, b \mid (b^a)^b = b^{a^2},\ (b^{a^2})^b = b^{a^3}\, \rangle$$

$\cong$ Strictly increasing PL homeomorphisms of $[0,1]$, that are differentiable except at finitely many dyadic rational numbers and such that all slopes are integer powers of 2.

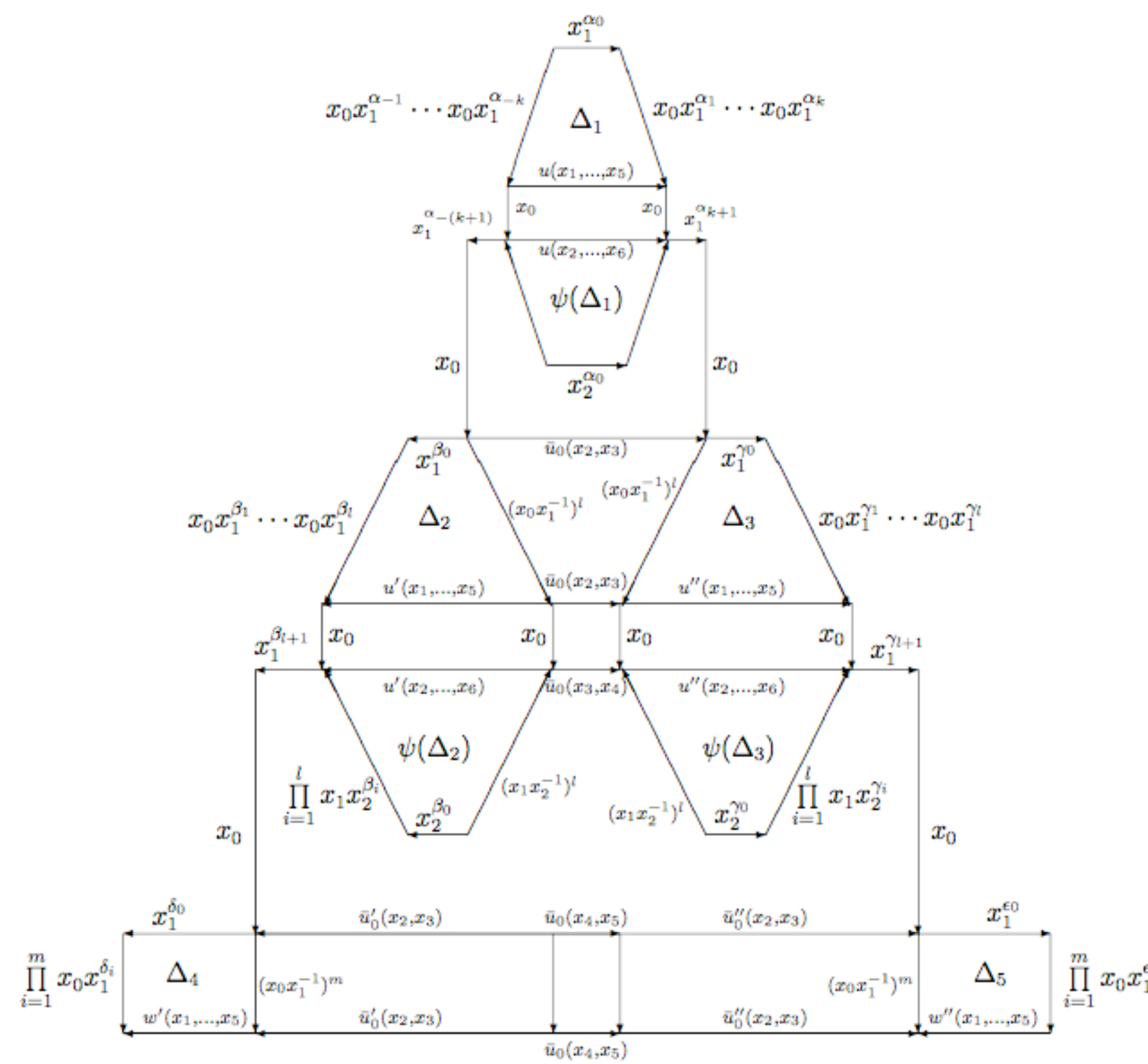
Theorem. [Guba] Thompson's group F enjoys a quadratic isoperimetric function.

Theorem. [Brin] Thompson's group F contains

$$\cdots(((\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}) \wr \mathbb{Z})\cdots \quad \text{and}$$

$$\cdots(\mathbb{Z} \wr (\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})))\cdots$$

as subgroups.

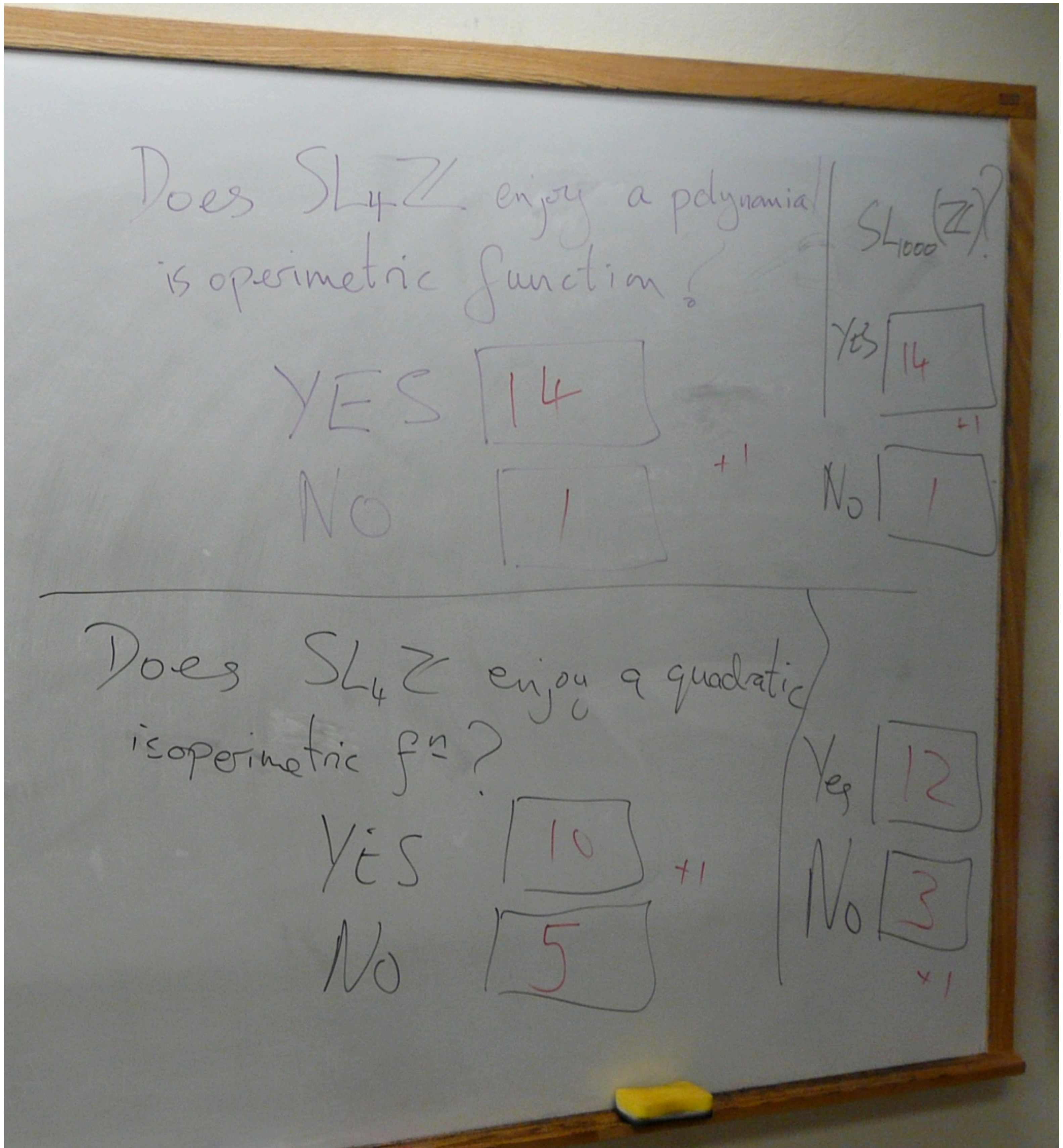


$\mathrm{SL}_n(\mathbb{Z})$

Thurston's Assertion. The Dehn function of $\mathrm{SL}_4(\mathbb{Z})$ is quadratic.

Gromov's Suggestion. The higher isoperimetric inequalities for $\mathrm{SL}_n(\mathbb{Z})$ concerning filling k–spheres by $(k+1)$–discs agree with those for Euclidean space for $k \leq n-3$.

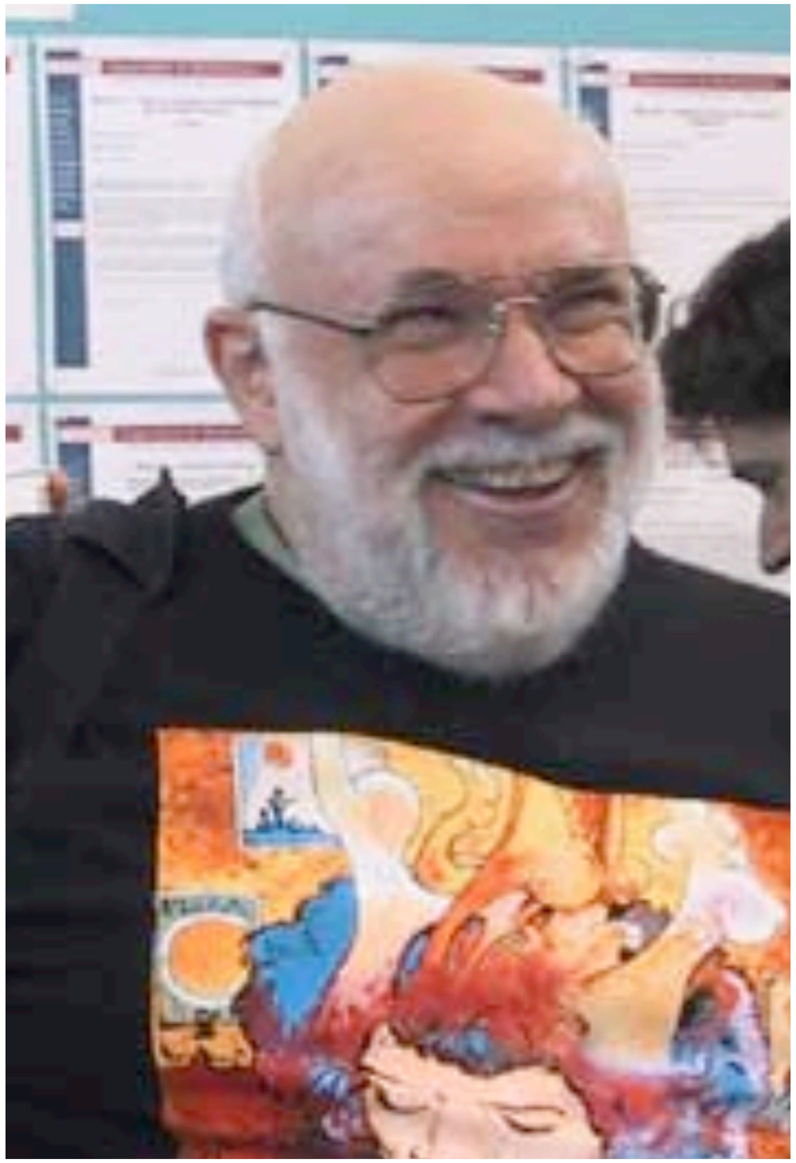
American Institute of Mathematics, 8th–12th September 2008


Does SL4Z enjoy a polynomial isoperimetric function?
YES 14
No 1
+1
SL1000(Z)?
Yes 14
+1
No 1
Does SL4Z enjoy a quadratic isoperimetric f^n?
YES 10
+1
No 5
Yes 12
No 3
+1

[Drutu] Every $\mathbb{Q}$–rank 1 lattice in a semisimple Lie group of $\mathbb{R}$–rank at least 3 is at most *asymptotically* quadratic. That is, $\forall \varepsilon > 0$, $\exists \ell_\varepsilon > 0$, $\forall \ell \geq \ell_\varepsilon$,

$$\mathrm{Area}(\ell) \leq \ell^{2+\varepsilon}.$$

Explicit examples: Hilbert modular groups $\mathrm{PSL}_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of a totally real field K with $[K : \mathbb{Q}] \geq 3$.



Steve M. Gersten

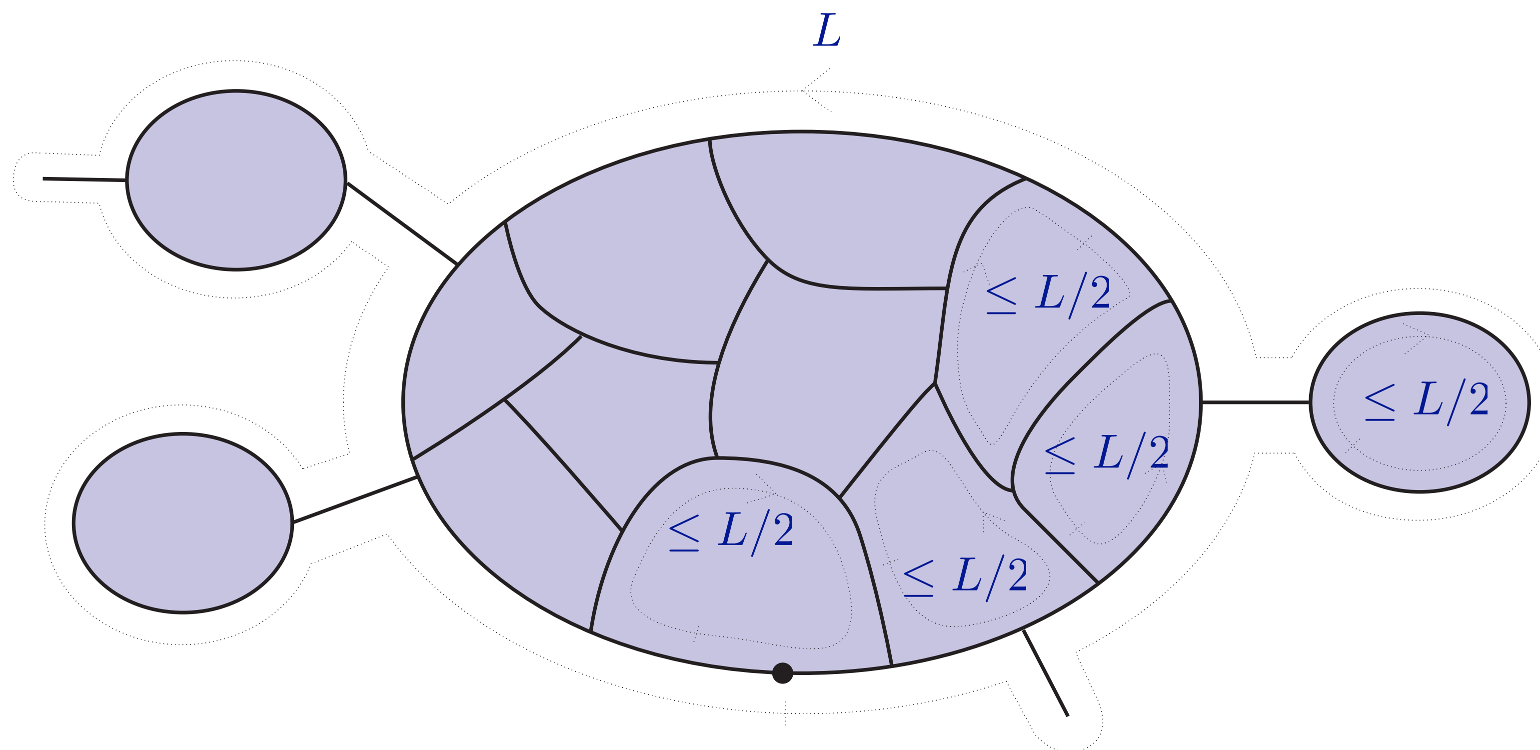
"I call this a zoo, because I am unable to see any pattern in this bestiary of groups. It would be striking if there existed a reasonable characterization of groups with quadratic Dehn functions, which was more enlightening than saying that they have quadratic Dehn functions."

Notes for a CRM Summer School on Groups held at Banff in 1996

Bestiary *n.*, a medieval collection of stories providing physical and allegorical descriptions of real or imaginary animals along with an interpretation of the moral significance each animal was thought to embody.

Asymptotic Cones

A graph enjoys the *loop–subdivision property* when there exists K such that every edge–loop of length $L \gg 0$ can be partitioned into $\leq K$ edge–loops of length $\leq L/2$.



Theorem. [Papasoglu] Groups with quadratic Dehn functions enjoy the loop–subdivision property.

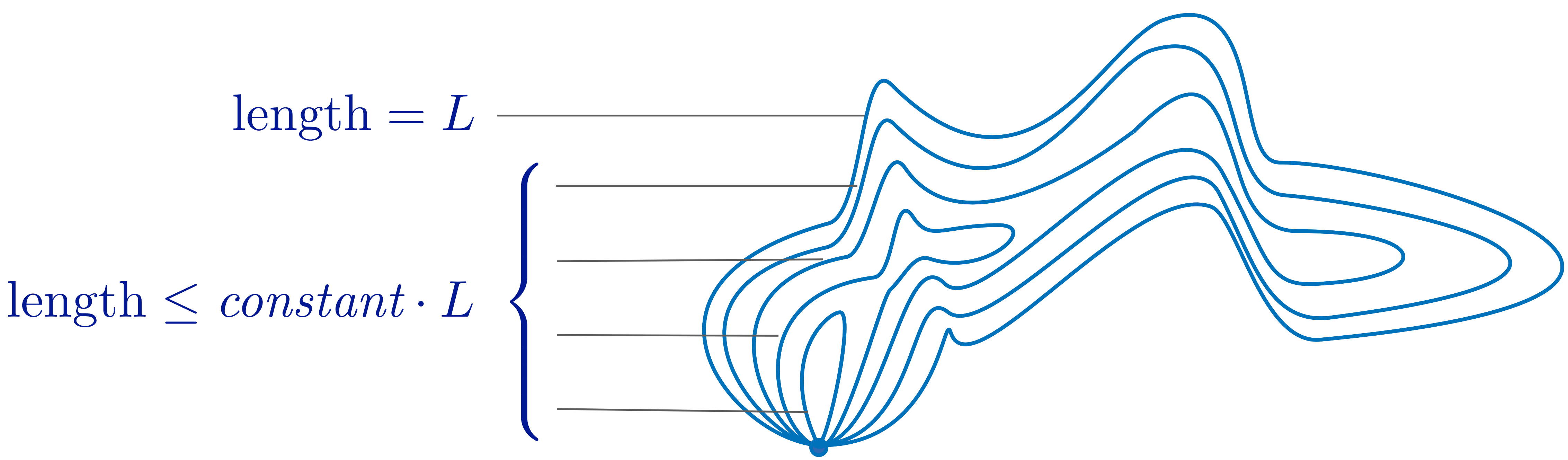
Theorem. [Gromov] A finitely generated group enjoys the loop–subdivision property if and only if all its asymptotic cones are simply connected.

Example. [Olshanskii–Sapir] The group

$$\langle\, a, b, c, k \;\mid\; [a,b], [a,c], b^{-1}kb = ka, c^{-1}kc = ka \,\rangle$$

has Dehn function $\simeq n^3$ but none of its asymptotic cones are simply connected. They claim that S–machines can be used to produce such an example with Dehn function $\simeq n^2 \log n$.

Theorem. [R.] The loop–subdivision property implies the *filling length function* is linear.*



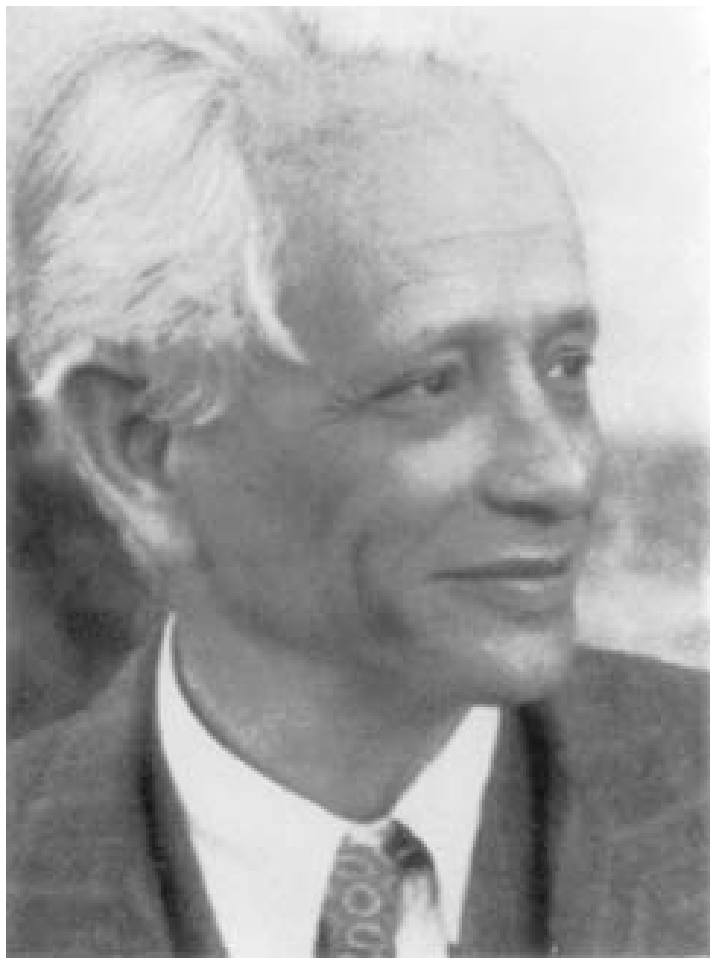
* This is a slight strengthening of a result of Papasoglu that there is a linear upper bound on the isodiametric function.

What next?

Open question. For each $n \geq 4$, is there a group which is of Type F_{n-1} but not of Type F_n and yet has a quadratic Dehn function?

Possible candidates:

Open question. What are the Dehn functions of the other Bieri–Stallings groups?


Max Dehn

The Conjugacy Problem

Given a group Γ with some generating set $\mathcal{A}$, find an algorithm which, on input two words on $\mathcal{A}^{\pm 1}$, declares whether or not they represent conjugate elements of Γ.

The Isomorphism Problem

Given a family of groups, find an algorithm which, on input two groups (e.g. given as finite presentations) of the family, declares whether or not they are isomorphic.

Conjecture. [Rips, Olshanskii–Sapir] Groups with quadratic Dehn function have solvable conjugacy problem.

Theorem. [Olshanskii–Sapir] There is a group with Dehn function $\simeq n^2 \log n$ but no algorithm to decide the conjugacy problem.

Open question: the isomorphism problem for groups with quadratic Dehn function.

cf. the isomorphism problem for $\mathrm{CAT}(0)$ groups is open,
... but is solved for torsion–free hyperbolic groups. [Sela]