Sandpiles and the Harmonic Model

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Spanning Forests on $\mathbb{Z}^2$

Consider the infinite graph Γ with vertex set $\mathbb{Z}^2$, and with nearest neighbours connected by edges. A connected subgraph $\Gamma' \subset \Gamma$ without loops is called a tree. A forest is a union of disjoint trees in Γ. A forest is spanning if it contains all vertices of Γ. A spanning forest is essential if all its trees are infinite.

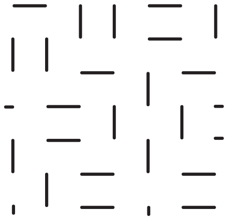
If $\mathcal{E}$ is the set of edges of Γ then the set X_F of all essential spanning forests in Γ can be viewed as a closed shift-invariant subset of $\{0,1\}^{\mathcal{E}}$. The shift-action σ of $\mathbb{Z}^2$ on X_F has topological entropy

$$h = \int_0^1 \int_0^1 \log\left(4 - 2\cos 2\pi s - 2\cos 2\pi t\right) ds\, dt, \qquad (\star)$$

and there is a unique shift-invariant probability measure μ on X_F with maximal entropy (Kasteleyn, 1961; Pemantle, 1991; Burton-Pemantle, 1993; Sheffield, 2006).

Dimers

Consider the even shift-action σ_D of $\mathbb{Z}^2$ on the space X_D of dimers which consists of all infinite configurations of exact pairings of elements in $\mathbb{Z}^2$ of the form



The topological entropy of σ_D is given by $(\star)$, and σ_D has a unique shift-invariant measure of maximal entropy.

Let $\Lambda \subset \mathbb{Z}^2$ be a nonempty finite set ($\Lambda \Subset \mathbb{Z}^2$).

- A configuration on Λ is an element of $\mathbb{N}^\Lambda$, where $\mathbb{N} = \{1, 2, 3, \dots\}$;
- A configuration $y \in \mathbb{N}^\Lambda$ is stable if $y_\mathbf{n} \le 4$ for every $\mathbf{n} \in \Lambda$;
- If $y \in \mathbb{N}^\Lambda$ is unstable at a site $\mathbf{n} \in \Lambda$ (i.e., if $y_\mathbf{n} > 4$), then the site $\mathbf{n}$ topples: $y \to y' = T_\mathbf{n} y$, where $y'_\mathbf{n} = y_\mathbf{n} - 4$, and $y'_\mathbf{m} = y_\mathbf{m} + 1$ for $\mathbf{m} \in \Lambda$ with $\|\mathbf{m} - \mathbf{n}\| = 1$. If $y_\mathbf{n} \le 4$ then $T_\mathbf{n} y = y$.

Lemma (Dhar, 1990). Put $\mathcal{T}_\Lambda = \lim_{n\to\infty} \prod_{\mathbf{n}\in\Lambda} T_\mathbf{n}$. Then $\mathcal{T}_\Lambda \colon \mathbb{N}^\Lambda \longrightarrow \mathbb{N}^\Lambda$ is well-defined, and $\mathcal{T}_\Lambda(y)$ is the stabilization of $y \in \mathbb{N}^\Lambda$.

We denote by $A_\mathbf{n} \colon y \mapsto A_\mathbf{n} y$ the addition of a single grain of sand at the location $\mathbf{n} \in \Lambda$ to a configuration $y \in \mathbb{N}^\Lambda$, and we write $\mathcal{A}_\mathbf{n}(y)$ for the stabilization of $A_\mathbf{n} y$.

Lemma (Dhar, 1990). The addition operators $\mathcal{A}_\mathbf{n}$, $\mathbf{n} \in \Lambda$, commute.

Addition of a grain of sand:

$$\boxed{\begin{matrix} 3 & 3 & 2 & 1 \\ 3 & \boxed{4} & 4 & 3 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 3 & 2 \end{matrix}} \xrightarrow{\text{addition}} \begin{matrix} 3 & 3 & 2 & 1 \\ 3 & \boxed{5} & 4 & 3 \\ 4 & 4 & 3 & 2 \\ 3 & 3 & 3 & 2 \end{matrix} \overset{\text{toppling}}{\rightsquigarrow} \begin{matrix} 3 & 4 & 2 & 1 \\ 4 & 1 & \boxed{5} & 3 \\ 4 & \boxed{5} & 3 & 2 \\ 3 & 3 & 3 & 2 \end{matrix}$$

$$\overset{\text{two topplings}}{\rightsquigarrow} \begin{matrix} 3 & 4 & 3 & 1 \\ 4 & 3 & 1 & 4 \\ \boxed{5} & 1 & \boxed{5} & 2 \\ 3 & 4 & 3 & 2 \end{matrix} \overset{\text{two topplings}}{\rightsquigarrow} \begin{matrix} 3 & 4 & 3 & 1 \\ \boxed{5} & 3 & 2 & 4 \\ 1 & 3 & 1 & 3 \\ 4 & 4 & 4 & 2 \end{matrix} \overset{\text{toppling}}{\rightsquigarrow} \boxed{\begin{matrix} 4 & 4 & 3 & 1 \\ 1 & 4 & 2 & 4 \\ 2 & 3 & 1 & 3 \\ 4 & 4 & 4 & 2 \end{matrix}}$$

Under addition and stabilization the stable configurations on Λ form a semigroup $\mathcal{S}_\Lambda$.

The group property

Definition: The set $\mathcal{R}_\Lambda \subset \mathcal{S}_\Lambda$ of recurrent configurations is the unique maximal subgroup of $\mathcal{S}_\Lambda$.

Description of $\mathcal{R}_\Lambda$: For every $E \subset \Lambda$ and $\mathbf{n} \in E$ we denote by $\mathsf{N}_E(\mathbf{n})$ the number of neighbours of $\mathbf{n}$ in E. Put

$$\mathcal{P}_E = \{v \in \mathcal{S}_\Lambda : v_{\mathbf{n}} > \mathsf{N}_E(\mathbf{n}) \text{ for at least one } \mathbf{n} \in E\}.$$

Then

$$\mathcal{R}_\Lambda = \bigcap\nolimits_{E \subset \Lambda} \mathcal{P}_E;$$

this is the characterization of $\mathcal{R}_\Lambda$ by the burning algorithm.

Note that $\pi_\Lambda(\mathcal{R}_{\Lambda'}) \subset \mathcal{R}_\Lambda$ whenever $\Lambda \subset \Lambda' \Subset \mathbb{Z}^2$.

Are the group operations in $\mathcal{R}_\Lambda$ and $\mathcal{R}_{\Lambda'}$ compatible? Unfortunately not!

Neutral Elements Of $\mathcal{R}_\Lambda$

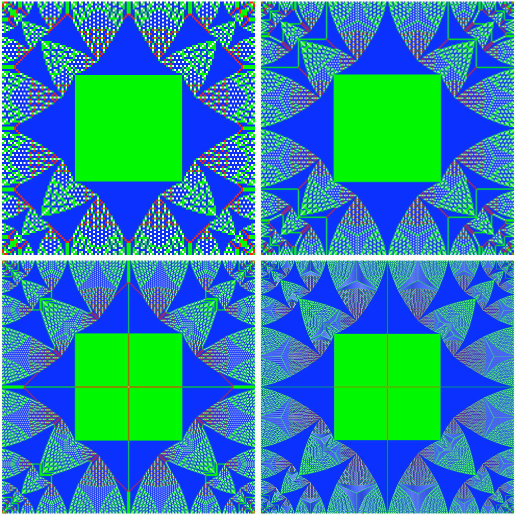


FIGURE 4. The identity element of the sandpile group of the $L \times L$ square grid for different values of L, namely $L = 128$ (upper left), 198 (upper right), 243 (lower left), and 521 (lower right). The color scheme is as follows: orange=0 chips, red=1 chip, green=2 chips, and blue=3 chips.

From Finite to Infinite Volume

Since $\pi_\Lambda(\mathcal{R}_{\Lambda'}) \subset \mathcal{R}_\Lambda$ whenever $\Lambda \subset \Lambda' \Subset \mathbb{Z}^2$ we can define a closed shift-invariant subset

$$\mathcal{R}_\infty = \{v \in \{1,2,3,4\}^{\mathbb{Z}^2} : \pi_\Lambda(v) \in \mathcal{R}_\Lambda \text{ for every } \Lambda \Subset \mathbb{Z}^2\},$$

called the 2-dimensional critical sandpile model (Bak-Tang-Wiesenfeld, 1988; Dhar, 1990).

- *What happens to addition on $\mathcal{R}_\Lambda$ as $\Lambda \nearrow \mathbb{Z}^2$? Is $\mathcal{R}_\infty$ still a group?*
- *If 'addition' can be defined on $\mathcal{R}_\infty$, what is the interaction between addition and the shifts $\sigma_{\mathcal{R}_\infty}^{\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^2$?*

The topological entropy of the shift action $\sigma_{\mathcal{R}_\infty}$ on $\mathcal{R}_\infty$ is given by $(\star)$ (cf. Dhar, 2006).

The Harmonic Model

Let $\alpha_{\mathcal{X}}$ be the shift-action of $\mathbb{Z}^2$ on the closed shift-invariant subgroup

$$\mathcal{X} = \Bigl\{(x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^2} : 4x_{\mathbf{n}} = x_{\mathbf{n}+(1,0)} + x_{\mathbf{n}-(1,0)} + x_{\mathbf{n}+(0,1)} + x_{\mathbf{n}-(0,1)} \text{ for all } \mathbf{n}\Bigr\}$$

of $\mathbb{T}^{\mathbb{Z}^2}$. Note that this is the linear recurrence relation on $\mathbb{T}^{\mathbb{Z}^2}$ defined by the Laurent polynomial $f = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1}$.

Theorem (Lind-S-Ward, 1990; Rudolph-S, 1995). $\alpha_{\mathcal{X}}$ is Bernoulli with entropy $(\star)$.

The Haar measure $\lambda_{\mathcal{X}}$ is the unique shift-invariant measure of maximal entropy on $\mathcal{X}$ (Lind-S-Ward, 1990).

The compact connected abelian group $\mathcal{X}$ (with Haar measure $\lambda_{\mathcal{X}}$ and shift-action $\alpha_{\mathcal{X}}$) is the *2-dimensional harmonic model*.

Connections between these models

All these models (spanning trees, dimers, sandpiles and the harmonic model) have the same topological entropy, and at least three of them have unique shift-invariant measures of maximal entropy which are Bernoulli (Burton-Pemantle, Rudolph-S).

Are there any connections between these models?

- The spanning tree model and the dimer model are ‘nicely’ isomorphic (Burton-Pemantle).
- For every finite set $\Lambda \subset \mathbb{Z}^2$ there is a natural bijection between restrictions to Λ of the spanning tree model and the sandpile model, but these maps are incompatible when Λ changes.

Sandpiles Cover The Harmonic Model

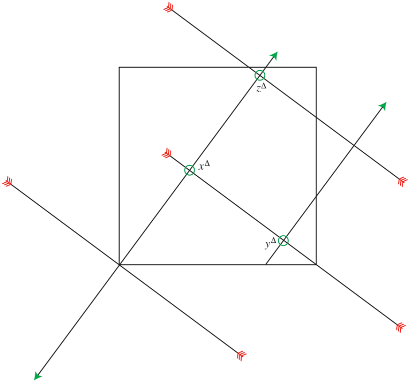
Theorem (Verbitskiy-S, 2009). There exists a continuous group homomorphism $\phi\colon \ell^\infty(\mathbb{Z}^2,\mathbb{Z}) \longrightarrow \mathcal{X}$ with the following properties.

- $\phi \circ \sigma_{\mathcal{R}_\infty} = \alpha_{\mathcal{X}} \circ \phi$.
- $\phi(\mathcal{R}_\infty) = \mathcal{X}$.
- ϕ sends every shift-invariant measure μ of maximal entropy on $\mathcal{R}_\infty$ to the normalized Haar measure $\lambda_{\mathcal{X}}$.

Conjecture: There exist a closed, α-invariant subgroup $\mathcal{Z} \subset \mathcal{X}$ and a continuous group homomorphism $\phi'\colon \ell^\infty(\mathbb{Z}^2,\mathbb{Z}) \longrightarrow \mathcal{X}/\mathcal{Z}$ with the following properties.

- $h(\alpha_{\mathcal{X}/\mathcal{Z}}) = h(\alpha_{\mathcal{X}})$. ✓
- $\phi' \circ \sigma_{\mathcal{R}_\infty} = \alpha_{\mathcal{X}/\mathcal{Z}} \circ \phi'$. ✓
- $\phi'(\mathcal{R}_\infty) = \mathcal{X}/\mathcal{Z}$. ✓
- ϕ' sends the unique? shift-invariant measure μ of maximal entropy on $\mathcal{R}_\infty$ to the Haar measure $\lambda_{\mathcal{X}/\mathcal{Z}}$.
- ϕ' is one-to-one μ-*a.e.* ?

Consider the hyperbolic automorphism $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We can draw its expanding and contracting subspaces:



The intersections of these two subspaces are the homoclinic points of α.

Maps Defined By Homoclinic Points

If x is one of these homoclinic points, then we can define a map $\xi_x \colon \ell^\infty(\mathbb{Z},\mathbb{Z}) \longrightarrow \mathbb{T}^2$ by

$$\xi_x(v) = \sum\nolimits_{n\in\mathbb{Z}} v_n \alpha^{-n} x, \ \ v = (v_n) \in \ell^\infty(\mathbb{Z},\mathbb{Z}).$$

The map ξ_x is equivariant:

$$\xi_x \circ \sigma = \alpha \circ \xi_x,$$

where σ is the shift $(\sigma v)_n = v_{n+1}$ on $\ell^\infty(\mathbb{Z},\mathbb{Z})$.

Symbolic Covers

In 1992–94 Vershik showed that the restriction of ξ_x to the two-sided beta-shift $X_\beta \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ of the large eigenvalue $\beta = \frac{1+\sqrt{5}}{2}$ of α is surjective: $\xi_x(X_\beta) = \mathbb{T}^2$.

Since X_β is the Golden Mean shift (which is of finite type), it is a Markov cover of α.

Subsequently it was shown that if one picks a good homoclinic point x of α, then ϕ_x is almost one-to-one, where good means that the orbit $\{\alpha^n x : n \in \mathbb{Z}\}$ generates the group $\Delta_\alpha(\mathbb{T}^2)$ of *all* homoclinic points of α. Such a homoclinic point is called fundamental.

This is essentially how the map $\psi \colon \mathcal{R}_\infty \longrightarrow \mathcal{X}$ is constructed: we construct homoclinic points of the harmonic model which decay sufficiently fast in the sense that their coordinates are summable.

L^1-Homoclinic Points Of The Harmonic Model

Let ν be the uniform probability measure on $\{(\pm 1, 0), (0, \pm 1)\} \subset \mathbb{Z}^2$. The random walk defined by ν is recurrent, hence $\mu = \sum_{k \geq 0} \nu^{*k}$ is purely infinite.

However, if we set $\mu_N(\{\mathbf{n}\}) = \sum_{k=0}^{N} \nu^{*k}(\{\mathbf{n}\})$ for all $\mathbf{n} \in \mathbb{Z}^2$, and if $\mu'_N(\{\mathbf{n}\}) = \mu_N(\{\mathbf{n}\}) - \mu_N(\{\mathbf{0}\})$, then $\mu' = \lim_{N \to \infty} \mu'_N$ is a negative σ-finite measure on $\mathbb{Z}^2$, called the Green's function of ν.

Consider the formal power series $F = \sum_{\mathbf{n} \in \mathbb{Z}^2} 4\mu'(\{\mathbf{n}\})\, u^{\mathbf{n}}$ with $u^{\mathbf{n}} = u_1^{n_1} u_2^{n_2}$ for all $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$. Then $F \cdot f = f \cdot F = 1$, where $f = 4 - u_1 - u_1^{-1} - u_2 - u_2^{-1}$.

Define $x^{\Delta} \in \mathbb{T}^{\mathbb{Z}^2}$ by $x^{\Delta}_{\mathbf{n}} = 4\mu'(\{\mathbf{n}\}) \pmod 1$. The last paragraph shows that $x^{\Delta} \in \mathcal{X}$. If is easy to check that x^{Δ} is *not* homoclinic. However, x^{Δ} generates *all* homoclinic points of $\mathcal{X}$: if $y \in \mathcal{X}$ is homoclinic, then $y = h(\alpha)(x^{\Delta})$ for some $h \in R_2 := \mathbb{Z}[u_1^{\pm 1}, u_2^{\pm 1}] = \mathbb{Z}(\mathbb{Z}^2)$, where $h(\alpha) = \sum_{\mathbf{n} \in \mathbb{Z}^2} h_{\mathbf{n}} \alpha^{\mathbf{n}}$ for every $h = \sum_{\mathbf{n} \in \mathbb{Z}^2} h_{\mathbf{n}} u^{\mathbf{n}} \in R_2$.

Theorem. A homoclinic point $y \in \mathcal{X}$ is L^1-homoclinic if and only if $y = h(\alpha)(x^{\Delta})$, where h lies in the ideal $I = ((1 - u_1), (1 - u_2))^3 \subset R_2$.

The Sandpile Model As A Cover Of $\mathcal{X}$

- For every $h \in I$ we obtain the L^1-homoclinic point $x_h = h(\alpha)(x^\Delta) \in \mathcal{X}$.
- This homoclinic point defines a shift-equivariant homomorphism from $\ell^\infty(\mathbb{Z}^2, \mathbb{Z})$ onto $\mathcal{X}$, whose restriction to $\mathcal{R}_\infty$ is surjective and entropy-preserving.
- The covering map $\phi' \colon \mathcal{R}_\infty \longrightarrow \mathcal{X}/\mathcal{Z}$ is obtained by combining all these homomorphisms $\xi_{x_h} \colon \mathcal{R}_\infty \longrightarrow \mathcal{X}$, $h \in I$, into a single map.