The Poisson boundary of certain Cartan-Hadamard manifolds of unbounded curvature

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Reference

Marc Arnaudon, Anton Thalmaier and Stefanie Ulsamer
Existence of non-trivial harmonic functions on Cartan-Hadamard manifolds of unbounded curvature
Mathematische Zeitschrift (2009) 41 pages

I. Geometric Background

"Geometry of a complex manifold influences its complex structure"

Question 1 (Wu 1983)

If a simply connected complete Kähler manifold has sectional curvature $\leq -c < 0$, is it biholomorphic to a bounded domain in $\mathbb{C}^n$?

Question 2 (Wu 1983)

If M is a simply-connected complete Riemannian manifold with sectional curvature $\leq -c < 0$, do there exist n bounded harmonic functions $(n = \dim M)$ which give global coordinates on M?

Note

- The second question may be seen as the Riemannian counterpart of the first question.
- Question 2 concerns richness of the space of bounded harmonic functions on negatively curved Riemannian manifolds

- Under the given assumptions, it is not known in general whether there exist non-trivial bounded harmonic functions at all.

Conjecture (Greene-Wu 1979)

Let M be a simply-connected complete Riemannian manifold of non-positive sectional curvature and $x_0 \in M$ such that

$$\mathrm{Sect}_x^M \le -c\, r(x)^{-2} \quad \text{for all } x \in M \setminus K$$

for some K compact, $c > 0$ and $r = \mathrm{dist}(x_0, \cdot)$.
Then M carries non-constant bounded harmonic functions.

II. Probabilistic formulation

Lemma

For a Riemannian manifold (M, g) the following two conditions are equivalent:

i) *There exist non-constant bounded harmonic functions on M.*

ii) BM *has non-trivial exit sets, i.e., if X is a Brownian motion on M then there exist open sets U in the 1-point compactification $\hat{M}$ of M such that*

$$\mathbb{P}\{X_t \in U \text{ eventually}\} \neq 0 \text{ or } 1.$$

More precisely

- Brownian motion X on M may be realized on the space $C(\mathbb{R}_+, \hat{M})$, equipped with the standard filtration $\mathscr{F}_t = \sigma\{X_s = \mathrm{pr}_s \,|\, s \le t\}$ generated by the coordinate projections.
- Let $\zeta = \sup\{t > 0 : X_t \in M\}$ be lifetime of X and let $\mathscr{F}_{\mathrm{inv}}$ denote the shift-invariant σ-field on $C(\mathbb{R}_+, \hat{M})$.
- Then there is a canonical isomorphism between the space $\mathscr{H}_{\mathrm{b}}(M)$ of bounded harmonic functions on M and the set $b\mathscr{F}_{\mathrm{inv}}$ of bounded $\mathscr{F}_{\mathrm{inv}}$-measurable random variables up to equivalence, given as follows:

$$\mathscr{H}_{\mathrm{b}}(M) \xrightarrow{\ \sim\ } b\mathscr{F}_{\mathrm{inv}}/\sim, \quad u \longmapsto \lim_{t\uparrow\zeta}(u \circ X_t).$$

(Bounded shift-invariant random variables are considered as equivalent, if they agree $\mathbb{P}_x$-a.e., for each $x \in M$.)

- The inverse map to this isorphism is given by taking expectations:

$$b\mathscr{F}_{\mathrm{inv}}/_{\sim} \ni H \longmapsto u \in \mathscr{H}_{\mathrm{b}}(M) \quad \text{where } u(x) := \mathbb{E}_x[H].$$

- In particular,

$$u(x) := \mathbb{P}_x\{X_t \in U \text{ eventually}\}$$

is a bounded harmonic function on M,
and *non-constant* if and only if U is a *non-trivial* exit set.

III. Generalities about CH manifolds

Let (M, g) be Cartan-Hadamard of dimension n
(i.e. simply-connected, metrically complete, $\mathrm{Sect}^M \leq 0$).

- Then $\exp_{x_0}\colon T_{x_0}M \xrightarrow{\sim} M$ and

$$\rho\colon \mathbb{R}^n \cong T_{x_0}M \xrightarrow{\sim} M.$$

  In particular $(M, g) \cong (\mathbb{R}^n, \rho^* g)$ is an isometric isomorphism.

- Thus

$$M \setminus \{x_0\} \cong]0, \infty[\times S^{n-1}$$

  and Brownian motion X on M may be decomposed into its radial and angular part,

$$X_t = (r(X_t), \vartheta(X_t))$$

  where $r(X_t) = \mathrm{dist}(x_0, X_t)$ and $\vartheta(X_t)$ takes values in S^{n-1}.

Geometric compactification of M

- We have

$$\bar{M} = M \mathbin{\dot\cup} M(\infty)$$

where $M(\infty) = \{\gamma : \mathbb{R} \to M \mid \gamma \text{ geodesics}\}/_\sim$

$$\gamma_1 \sim \gamma_2 \Longleftrightarrow \limsup_{t\to\infty} \operatorname{dist}(\gamma_1(t), \gamma_2(t)) < \infty$$

- Consequently,

  M diffeomorphic to the open ball in $\mathbb{R}^n$
  $\bar{M}$ homeomorphic to the closed ball in $\mathbb{R}^n$

- In particular,

$$S_\infty(M) := M(\infty) \equiv \{\gamma(\infty) \mid \gamma \text{ geodesics}\}$$

is an $(n-1)$-sphere ("*horizon at* ∞")

Dirichlet problem at infinity

- Given a continuous function

$$f\colon S_\infty(M) \to \mathbb{R}$$

the *Dirichlet problem at infinity* is to find a harmonic function $u\colon M \to \mathbb{R}$ which extends continuously to $S_\infty(M)$ and there coincides with the given function f, i.e.,

$$u|S_\infty(M) = f.$$

The Dirichlet problem at infinity is called *solvable* if this is possible for every such function f.

IV. Brownian motion on CH manifolds

J.-J. Prat (1975), Y. Kiefer (1976), D. Sullivan (1983)

Let (M, g) be Cartan-Hadamard of dimension n such that

$$-a^2 \leq \mathrm{Sect}^M \leq -b^2 < 0.$$

- Then a.s.,

$$\left.\begin{array}{l} r(X_t) \to \infty \\ \vartheta(X_t) \text{ converges on } S^{n-1} \end{array}\right\} \text{ as } t \to \infty.$$

- Denote $\Theta_\infty := \lim_{t\to\infty} \vartheta(X_t)$ and $\mu_x := \mathbb{P}_x \circ \Theta_\infty^{-1}$. Then

$$u(x) = \mathbb{P}_x\{\Theta_\infty \in U\} \equiv \mu_x(U) \quad \text{is harmonic on } M.$$

Note that for the Poisson hitting measure: $\mu_x \sim \mu_y$

Theorem (Sullivan 1983)

- *The harmonic measure class on $S_\infty(M)$ is positive on each non-void open set.*
- *In fact, if x_i in M converges to x_∞ in $S_\infty(M)$, then the Poisson hitting measures μ_{x_i} tend weakly to the Dirac mass at x_∞.*

- **Corollary** Solvability of the *Dirichlet problem at infinity*

  For $f \in C(S_\infty(M))$ let $u(x) = \mathbb{E}_x\big[f(\Theta_\infty)\big]$.
  Then

$$u \in C(\bar{M}), \quad \Delta u = 0 \text{ on } M \quad \text{and} \quad u|S_\infty(M) = f.$$

- Moreover, any bounded harmonic function on M comes from a solution of the DP at ∞ (for some bounded measurable f).

Theorem (Anderson 1983)

Let (M, g) be a Cartan-Hadamard manifold of dimension n, whose sectional curvatures satisfy

$$-a^2 \leq \mathrm{Sect}_x^M \leq -b^2$$

for all $x \in M$. Then the linear mapping

$$P\colon L^\infty(S_\infty(M), \mu) \to \mathscr{H}_\mathrm{b}(M),$$
$$f \mapsto P(f), \quad P(f)(x) := \int_{S_\infty(M)} f d\mu_x$$

is a norm-nonincreasing isomorphism onto $\mathscr{H}_\mathrm{b}(M)$.

- In particular, $\mathscr{F}_\mathrm{inv} = \sigma(\Theta_\infty)$.

Probabilistic conditions for the solvability of the DP at ∞:

- $r(X_t) \to \infty$ for the radial part of BM
- *convergence of the angular part* $\vartheta(X_t)$;
  this defines entrance measures μ_x on the sphere at ∞
- $\mu_{x_n} \to \delta_{x_\infty}$ in probability, as $x_n \to x_\infty \in S_\infty(M)$

Resumé The upper curvature bound is needed to guarantee a certain rate of escape for BM, whereas the lower curvature bound is necessary to control the angular oscillations.

- The constant curvature bounds can be relaxed:
  Hsu-March (1985), Borbély (1993), Elton P. Hsu (2003), Vähäkangas (2007), Holopainen-Vähäkangas (2007)

Theorem (M. Arnaudon, Elton P. Hsu, A.Th.)

Let M be a simply-connected complete Riemannian manifold of non-positive sectional curvature such that

$$\mathrm{Sect}_x^M \le -\frac{c}{r^2(x)} \quad \forall\, x \in M \setminus K$$

for some K compact, $c > 0$ and $r = \mathrm{dist}(x_0, \cdot)$.
Suppose that there exists $C > 1$ such that for each $x \in M \setminus K$ and all radial planes E, $E' \subset T_xM$

$$|\mathrm{Sect}_x^M(E)| \le C\,|\mathrm{Sect}_x^M(E')|.$$

Then the DP at ∞ is solvable.

V. Role of lower curvature bounds

Choi (1984) "*convex conic neighbourhood condition*"

(C) *Every two distinct points* $x, y \in S_\infty(M)$ *can be separated in* $\bar{M}$ *by convex sets.*

Condition (C) + $\mathrm{Sect}^M \leq -1$

$\Longrightarrow$ Solvability of the DP at ∞

Ancona (1994) constructed a complete, simply connected Riemannian manifold M of dimension 3, with $\mathrm{Sect}^M \leq -1$, and a point $\infty_M \in S_\infty(M)$ such that

(i) $\mathrm{BM}(M, g)$ *has a.s. infinite lifetime*;

(ii) *With probability* 1, $\mathrm{BM}(M, g)$ *exits from* M *at* ∞_M.

In addition, the convex hull of any neighbourhood of ∞_M in $\bar{M}$ contains all of M.

Borbély (1998) Construction of a similar example

Consequence

$$\left.\begin{array}{l}\text{Suppose that } f \in C(S_\infty(M)) \text{ and } u \in C(\bar{M}),\\ \text{such that } \Delta u = 0 \text{ on } M,\ u|S_\infty(M) = f\end{array}\right\} \Longrightarrow u \equiv \text{const.}$$

Indeed, $u(x) = \mathbb{E}_x\big[f(X_\infty)\big] = f(\infty_M) \quad \forall\, x \in M.$

Question Existence of non-trivial exit sets for BM on M?

VI. Example

Consider $\mathbb{H}^2 \equiv \mathbb{H}^2(-1) = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

Let $L = \{(0,y) : y > 0\}$ and let H be one component of $\mathbb{H}^2 \setminus L$.

Definition $\boxed{M := (H \cup L) \times_g S^1}$

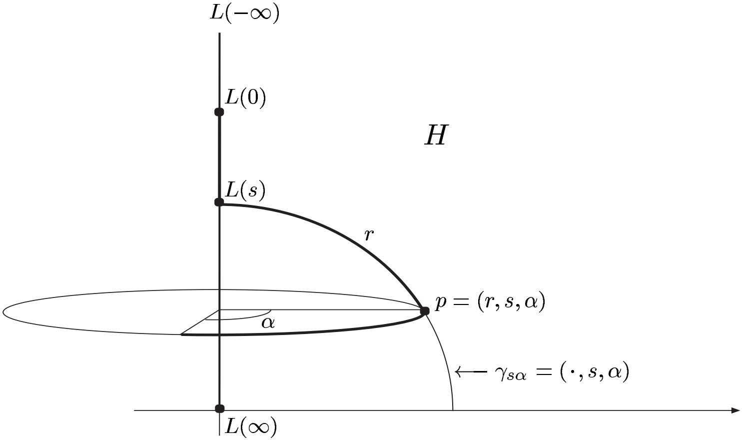
with $g\colon H \cup L \to \mathbb{R}_+$ C^∞ appropriate.

In terms of Fermi coordinates (r, s, α) for M:

$$\boxed{ds_M^2 = dr^2 + h(r)\, ds^2 + g(r,s)\, d\alpha^2}$$

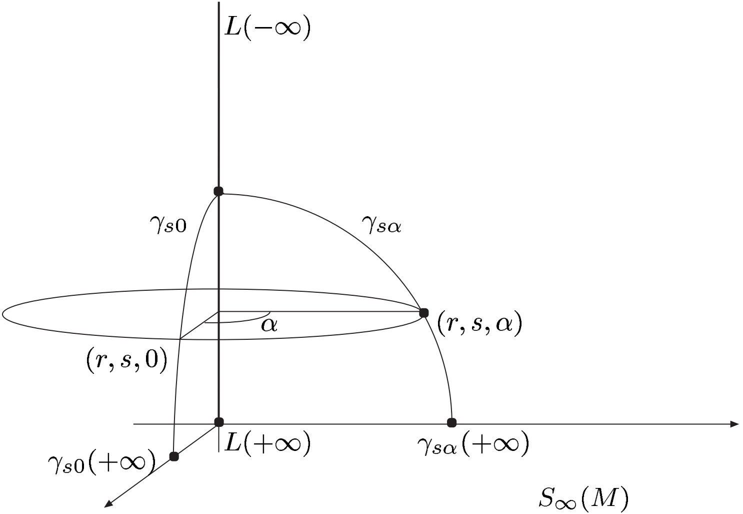
where $h(r) = \cosh^2(r)$ and $g(r,s) = \sinh^2(r)$, $r \le \frac{1}{10}$

Note $(\ell, \alpha_1) \sim (\ell, \alpha_2)$, $\quad \ell \in L$, $\alpha_i \in S^1$



Fermi coordinates (r, s, α) for M

$$S_\infty(M) = \{\gamma_{s\alpha}(\infty) \mid s \in \mathbb{R},\ \alpha \in [0, 2\pi[\} \cup \{L(\pm\infty)\}$$



Sphere at Infinity $S_\infty(M)$

Theorem

There exists a smooth wedge function

$$g\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$$

such that

(i) (M, ds_M^2) *is Cartan-Hadamard with* $\mathrm{Sect}^M \le -1$

(ii) $X_t \to L(+\infty)$ *a.s. for each Brownian motion* X *on* M

- More precisely, we have

  $$X_t = (R_t, S_t, A_t) \quad \text{with} \quad \begin{aligned} R_t &\to +\infty \ \text{a.s.} \\ S_t &\to +\infty \ \text{a.s.} \\ A_t &\to \quad A_\zeta \ \text{a.s.,} \end{aligned}$$

  where A_ζ is a non-trivial random variable taking values in S^1.
- The support of $\mathbb{P}_x \circ (A_\zeta)^{-1}$ equals S^1 for any $x \in M$.

- **Consequence** For each $f \in C(S^1; \mathbb{R})$ non-constant,

$$h(x) = \mathbb{E}_x[f(A_\zeta)]$$

is a non-trivial bounded harmonic function on M.

- **Interpretation** The variable A_ζ gives the asymptotic direction on the sphere at ∞ along which BM approaches the point

$$\infty_M = L(+\infty) \in S_\infty(M).$$

- **Question** Are there other non-trivial bounded harmonic functions?

- Consider BM in the specified Fermi coordinates

$$\begin{cases} dR = F^1(R,S)\,dt + dM^1 \\ dS = F^2(R,S)\,dt + dM^2 \\ dA = \qquad\qquad\qquad dM^3 \end{cases}$$

- After a change of time

$$\begin{cases} d\hat{R} = \qquad\qquad dt + d\hat{M}^1 \\ d\hat{S} = f(\hat{R},\hat{S})\,dt + d\hat{M}^2 \\ d\hat{A} = \qquad\qquad\qquad d\hat{M}^3 \end{cases}$$

where the martingale parts $\hat{M}^i_t$ converge as $t \to \zeta$.

- *Idea* Consider in the (r, s)-plan

$$\begin{cases} dr_t = dt \\ ds_t = f(r_t, s_t)\, dt \end{cases} \quad \text{i.e.} \quad \begin{cases} \dot{r}_t = 1 \\ \dot{s}_t = f(t, s_t). \end{cases}$$

- *Then it seems reasonable to expect the following:*
  For $t \gg 0$,

$$(\hat{R}_t, \hat{S}_t,) \approx (r_t, s_t)$$

  where (r_t, s_t) is an integral curve of the vector field

$$V = \frac{\partial}{\partial r} + f(r, s) \frac{\partial}{\partial s}.$$

- Moreover, there is a good approximation $q = q(r)$ such that

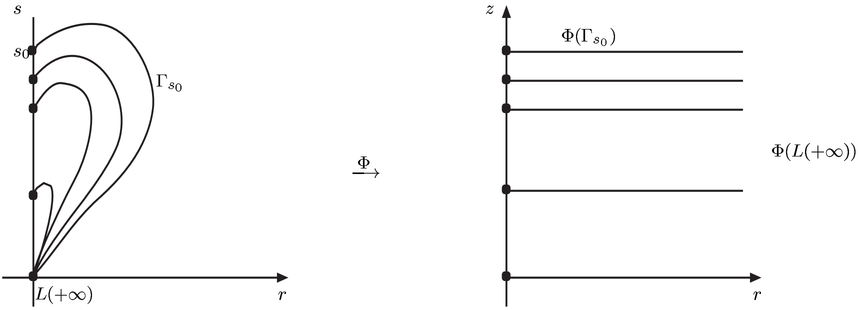
$$f(r, s) \approx q(r) \quad \text{for large } r.$$

- Let $\Gamma_{s_0}(t) = \left(t, s_0 + \int_0^t q(u)\,du\right)$ and denote

$$\Gamma_{s_0} := \{\Gamma_{s_0}(t) : t \geq 0\}.$$

- Then $(\Gamma_{s_0})_{s_0 \in \mathbb{R}}$ defines a foliation of H.
- Consider the following change of coordinates:

$$\begin{aligned} \Phi :\ & \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}, \\ & (r, s) \to \left(r, s - \int_0^r q(u)\,du\right) =: (r, z) \end{aligned}$$



The coordinate transformation Φ

Consider the BM in the new coordinates:

$$X = (R, S, A) \longmapsto X = (R, Z, A)$$

where $Z_t = S_t - \int_0^{R_t} q(u)\, du$.

Theorem

- *We have, a.s., as $t \to \zeta$,*

$$\begin{cases} R_t \to \infty \\ Z_t \to Z_\zeta \in \mathbb{R} \\ A_t \to A_\zeta \in S^1 \end{cases}$$

- *The induced measures*

$$\begin{array}{ll} \mathbb{P}_x \circ (Z_\zeta)^{-1} \ \text{ on } \mathbb{R}, & \textit{respectively,} \\ \mathbb{P}_x \circ (A_\zeta)^{-1} \ \text{ on } S^1, & \end{array}$$

have full support.

Consequences

- For each $f \in C_{\mathrm{b}}(\mathbb{R} \times S^1)$, $f \neq \mathrm{const}$,

$$u(x) = \mathbb{E}_x[f(Z_\zeta, A_\zeta)]$$

  defines a non-trivial bounded harmonic function on M.
- *Non-trivial question* $\mathscr{F}_{\mathrm{inv}} = \sigma(Z_\zeta, A_\zeta)$?
- *Answer*: Yes!
- *Method of proof*
  - Time reversal arguments

    É. Pardoux (1986) *Grossissement d'une filtration et retournement du temps d'une diffusion*.
  - Coupling arguments for the time-reversed process
  - 0/1 law for the time-reversed process
  - etc

Theorem (M. Arnaudon, A. Th., S. Ulsamer)

Let $\mathscr{B}(\mathbb{R} \times S^1; \mathbb{R})$ be the set of bounded measurable functions on $\mathbb{R} \times S^1$, with the equivalence relation $f_1 \simeq f_2$ if $f_1 = f_2$ Leb.-a.e. The map

$$\begin{aligned}(\mathscr{B}(\mathbb{R} \times S^1; \mathbb{R})/\simeq) &\longrightarrow \mathscr{H}_b(M)\\ f &\longmapsto \big(x \mapsto \mathbb{E}\left[f\left(Z_\zeta(x), A_\zeta(x)\right)\right]\big)\end{aligned}$$

is one to one. The inverse map is given as follows. For $x \in M$, letting $K(x,\cdot,\cdot)$ be the density of $(Z_\zeta(x), A_\zeta(x))$ with respect to the Lebesgue measure on $\mathbb{R} \times S^1$, for all $h \in \mathscr{H}_b(M)$ there exists a unique $f \in \mathscr{B}(\mathbb{R} \times S^1; \mathbb{R})/\simeq$ such that

$$\forall x \in M, \quad h(x) = \int_{\mathbb{R}\times S_1} K(x,z,a) f(z,a)\, dz da.$$

Moreover, for $x \in M$, the kernel $K(x,\cdot,\cdot)$ is a.e. strictly positive.

Comments

- The space $\mathscr{H}_{\mathrm{b}}(M)$ is as rich as in the pinched curvature case

$$-a^2 \leq \mathrm{Sect}^M \leq -b^2 < 0,$$

where we have

$$\mathscr{F}_{\mathrm{inv}} = \sigma(\Theta_\infty).$$

- The harmonic measure has a density (Poisson kernel) with respect to the Lebesgue measure.
  [Note that in the pinched curvature case the harmonic measure may well be singular with respect to the surface measure on the sphere at infinity.]
- To see the full Poisson boundary in our example the point at infinity needs to be blown up in a non-trivial way.
- On any neighbourhood in M of the point $\infty_M \in S_\infty(M)$, a bounded harmonic function on M attains each value between its global minimum and global maximum.

Outlook

- *Observation* The above example can be modified in such a way that, with probability 1, every point on $S_\infty(M)$ is a cluster point of X_t (when $t \to \infty$).
- *Non-trivial question* In this case, do we have

$$\mathscr{H}_{\mathrm{b}}(M) = \{\text{constants}\} \ ?$$