Graz University of Technology

Department of Mathematical Structure Theory

Diskrete Stochastik und Informationstheorie – 2 Apr 2014

Exercise 11. (Law of total expectation) Let X be a discrete random variable satisfying $\mathbb{E}(|X|) < \infty$ and Y be another random variable taking values in $\{y_1, \dots, y_n\}$, both over the same probability space. Show that

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X|Y = y_i) \cdot \mathbb{P}(Y = y_i).$$

Exercise 12. Let X be a discrete random variable taking values in $\mathbb{N}$. Show that $\mathbb{E}(X) < \infty$ if and only if $\sum_{n\geq 1} \mathbb{P}[X > n] < \infty$.

Exercise 13. Let $(X_n)_{n\geq 1}$ be a sequence of nonnegative discrete random variables. Assume $\mathbb{E}(\sum_{n=1}^{\infty} X_n) < \infty$. Show that $\lim_{n\to\infty} X_n = 0$ almost surely.

(Hint: Set $Y = \sum_{n=1}^{\infty} X_n$ and use the previous example and the Borel-Cantelli lemma.)

Exercise 14. Consider the following two-stage experiment: We first throw a die. If the result is 1, we draw a ball from urn A; otherwise, we draw from urn B. Urn A contains 2 black balls and 3 white balls; urn B contains 3 red balls and 4 blue balls. Let X be the color of the drawn ball. Calculate the entropy of X.

Exercise 15. A fair coin is flipped until the first head occurs. Let X denote the number of flips required.

(a) Calculate the entropy $H(X)$ in bits.

(Hint: Think about $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ and deriving both sides of this equation to evaluate the series occurring in the calculation.)

(b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions to determine the value of X. Compare $H(X)$ to the expected number of questions required to determine the value of X.