Graz University of Technology

Department of Mathematical Structure Theory

Diskrete Stochastik und Informationstheorie – 28 May 2014

Exercise 30. Let

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2}, \\ 2 & \text{with probability } \frac{1}{3}, \\ 3 & \text{with probability } \frac{1}{6}. \end{cases}$$

Let $X_1, X_2, \ldots$ be drawn independently and identically distributed (i.i.d.) according to the distribution of X. Find the limiting behavior of $(X_1 \cdot X_2 \cdots X_n)^{1/n}$ as $n \to \infty$.
Hint: How could the target limit be transformed/rewritten to apply the ergodic theorem or the law of large numbers?

Exercise 31. Let X be the outcome of a biased coin flip with $\mathcal{X} = \{0, 1\}$, where $\mathbb{P}[X = 1] = p > \frac{1}{2}$ and $\mathbb{P}[X = 0] = 1 - p$. Let $(X_i)_{1 \le i \le n}$ be an i.i.d. sequence of such coin flips, and denote a particular outcome by $x^{(n)} = (x_1, \ldots, x_n) \in \mathcal{X}^n$. Define the typical set $A_\varepsilon^{(n)}$ by

$$A_\varepsilon^{(n)} = \Big\{ (x_1, \ldots, x_n) \in \mathcal{X}^n : \Big| -\frac{1}{n} \log p_n(x_1, \ldots, x_n) - H(X) \Big| \le \varepsilon \Big\}.$$

(a) Show that $x^{(n)} \in A_\varepsilon^{(n)}$ if and only if

$$p - \frac{\varepsilon}{\left|\log \frac{1-p}{p}\right|} \le \frac{r}{n} \le p + \frac{\varepsilon}{\left|\log \frac{1-p}{p}\right|},$$

where r is the number of "1"s in the string $x^{(n)}$.

(b) Calculate $\mathbb{P}(A_\varepsilon^{(n)})$ for $p = 0.7$, $\varepsilon = 10^{-2}$ and $n = 100$, $n = 1000$ and $n = 10000$.

(c) Calculate $\mathbb{P}(A_\varepsilon^{(n)})$ for $p = 0.99$, $\varepsilon = 10^{-2}$ and $n = 100$. How many strings are there in $A_\varepsilon^{(n)}$? Show that it is possible to give a coding for elements in $\mathcal{X}^n$ such that the expected string length is $\mathbb{E}(\ell(x^{(n)})) \le 68$ instead of the expected 100 bits for trivial binary coding.

(d) How many strings are there in $A_\varepsilon^{(n)}$ for $p = \frac{1}{2}$ and arbitrary ε, n? Does typical set encoding make sense in this case?

(e) Let $p > \frac{3}{4}$ and $\varepsilon = \frac{1}{n}$. Calculate $\lim_{n\to\infty} \mathbb{P}(A_\varepsilon^{(n)})$.
Hint: Use Stirling's formula to approximate the factorials.

Exercise 32. Let X be a random variable with values in $\mathcal{X} = \{1, 2, \ldots, m\}$ and distribution $p(x)$, and let $(X_i)_{i \ge 0}$ be i.i.d. random variables with the same distribution. Denote $\mu = \mathbb{E}(X)$ and $H = H(X) = -\sum p(x) \log p(x)$. For $\varepsilon > 0$, consider the sets

$$A^{(n)} = \Big\{ (x_1, \ldots, x_n) \in \mathcal{X}^n : \Big| -\frac{1}{n} \log p_n(x_1, \ldots, x_n) - H \Big| \le \varepsilon \Big\},$$

$$B^{(n)} = \Big\{ (x_1, \ldots, x_n) \in \mathcal{X}^n : \Big| \frac{1}{n} \sum_{i=1}^{n} x_i - \mu \Big| \le \varepsilon \Big\}.$$

(a) Does $\mathbb{P}[(X_1, \ldots, X_n) \in A^{(n)}] \to 1$ for $n \to \infty$?

(b) Does $\mathbb{P}[(X_1, \ldots, X_n) \in A^{(n)} \cap B^{(n)}] \to 1$ for $n \to \infty$?

(c) Show that $|A^{(n)} \cap B^{(n)}| \le 2^{n(H+\varepsilon)}$ for all n.

(d) Show that $|A^{(n)} \cap B^{(n)}| \ge (\frac{1}{2}) 2^{n(H-\varepsilon)}$ for all n sufficiently large.