Graz University of Technology

Department of Mathematical Structure Theory

Diskrete Stochastik und Informationstheorie – 4 June 2014

Exercise 33. Let $\mathcal{X} = \{a, b, c, d\}$, and let $X : \Omega \to \mathcal{X}$ be a random variable with

$$\mathbb{P}[X = a] = \frac{3}{8}, \qquad \mathbb{P}[X = b] = \frac{2}{8}, \qquad \mathbb{P}[X = c] = \frac{2}{8}, \qquad \mathbb{P}[X = d] = \frac{1}{8}.$$

The elements of $\mathcal{X}$ are encoded as follows:

$$C(a) = 00, \qquad C(b) = 01, \qquad C(c) = 11, \qquad C(d) = 001\,.$$

(a) Is the code C (i) non-singular, (ii) prefix-free, (iii) uniquely decodable?

(b) Calculate the entropy $H(X)$ and the expected length $\mathbb{E}(\ell(C))$.

(c) Give a better code for this random variable (prefix-free, shorter expected length).

Exercise 34. Let $\mathcal{X} = \{a, b, c, d, e\}$.

(a) Give an example of a prefix code $C : \mathcal{X} \to \{0, 1\}^*$ such that

$$\ell(C(a)) = 1, \qquad \ell(C(b)) = 3, \qquad \ell(C(c)) = \ell(C(d)) = 4, \qquad \ell(C(e)) = 5.$$

(b) Show that the Kraft inequality is a strict inequality for this code, i.e.,

$$\sum_{x \in \mathcal{X}} D^{-\ell(C(x))} < 1.$$

(c) Give an example of a sequence in $\{0, 1\}^*$ which cannot be decoded (in the sense that it is not a prefix of a valid encoded sequence, i.e., it cannot be extended by any suffix to become a valid codeword of the extension code).

(d) Give an example of a prefix code $C : \mathcal{X} \to \{0, 1\}^*$ such that any sequence in $\{0, 1\}^*$ can be decoded (i.e., any sequence is the prefix of a valid encoded sequence). Is the Kraft inequality a strict inequality for the new code?

Exercise 35 (Code from Kraft inequality)**.** Let $\mathcal{X} = \{a, b, c, d, e, f, g, h\}$.

(a) Use the algorithm from the proof of the Kraft inequality to design a prefix code $C : \mathcal{X} \to \{0, 1\}^*$ such that

$$\begin{aligned} &\ell(C(a)) = \ell(C(b)) = \ell(C(c)) = 2\,, \\ &\ell(C(d)) = 3\,, \\ &\ell(C(e)) = \ell(C(f)) = \ell(C(g)) = \ell(C(h)) = 5\,. \end{aligned}$$

(b) Give a probability distribution for a random variable $X : \Omega \to \mathcal{X}$ such that this coding is optimal and compare the code's expected length with the variable's entropy.