Diskrete Stochastik und Informationstheorie – 25 Jun 2014

Exercise 45. Consider the discrete channel $Y = X+Z \mod 11$, where $X \in \mathcal{X} = \{0,1,\dots,10\}$ and Z is a random noise independent of X with $\mathbb{P}[Z=1] = \mathbb{P}[Z=2] = \mathbb{P}[Z=3] = \frac{1}{3}$.

(a) What does the transition matrix of the channel look like?

(b) Show that $I(X;Y) \le \log_2 \frac{|\mathcal{X}|}{3}$.

(c) Give the probability distribution of X that maximizes $I(X;Y)$ to obtain the channel capacity. What is the corresponding distribution of Y?

Exercise 46. Let $\mathcal{X} = \mathcal{Y} = \{0,1\}$. The transition probabilities are given by

$$p(0|0) = 1, \quad p(1|0) = 0, \quad p(0|1) = p(1|1) = \frac{1}{2}.$$

(a) Is this channel weakly symmetric?

(b) Show that if $\mathbb{P}[X=1] = q$, then $I(X;Y)$ is given by $I(X;Y) = H(\frac{q}{2}, 1-\frac{q}{2}) - q$.

(c) Find the channel capacity.

Hint: Show that $q^* = \frac{2}{5}$ maximizes the above function (by determining the roots of the function's derivation).

Exercise 47. Consider two independent discrete channels $(\mathcal{X}_1, \mathcal{P}_1, \mathcal{Y}_1)$ and $(\mathcal{X}_2, \mathcal{P}_2, \mathcal{Y}_2)$. We construct a new channel $\mathcal{C}$ from $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ will be transmitted in parallel (at the same time): x_1 is mapped to some $y_1 \in \mathcal{Y}_1$ and x_2 is mapped to some $y_2 \in \mathcal{Y}_2$. Show that the capacity of this joint channel is

$$\mathrm{Cap}(\mathcal{C}) = \mathrm{Cap}(\mathcal{C}_1) + \mathrm{Cap}(\mathcal{C}_2).$$

Exercise 48. Let X and Y be two random variables with the following joint distribution:

$Y \backslash X$	0	1
a	$1/2$	$1/6$
b	$1/12$	$1/4$

Suppose X is the input and Y the output of a channel, and only Y can be observed. We want to find a good estimator $\hat{X}(y)$ of the value x. The estimator can be probabilistic, i.e., an observed value $y \in \{a,b\}$ is mapped to an estimate $\hat{x} \in \{0,1\}$ with probability $p_{y,\hat{x}}$.

(a) Argue that $(X, Y, \hat{X})$ is a Markov triple if $\hat{X}$ is estimated as above.

(b) Why is the corollary of Fano's inequality useless for bounding $\mathbb{P}[\hat{X}(Y) \neq X] = p_{\text{error}}$ in this example?

(c) Show that the best estimator of x (minimizing $\mathbb{P}[\hat{X}(Y) \neq X]$) is

$$\hat{X}(y) = \begin{cases} 0 & y = a \\ 1 & y = b \end{cases}$$

Hint: Use the Markov property of $X \mapsto Y \mapsto \hat{X}$ in

$$\mathbb{P}[\hat{X}(Y) \neq X] = \sum_{\substack{\hat{x},x,y \\ \hat{x} \neq x}} \mathbb{P}[\hat{X} = \hat{x}, Y = y, X = x].$$