Convolution semigroups with linear Jacobi parameters

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Jacobi parameters.

$\mu =$ measure with finite moments,

$$m_i = \int x^i \, d\mu(x) < \infty.$$

Jacobi parameters

$$\mu \leftrightarrow J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \ldots \\ \gamma_0, & \gamma_1, & \gamma_2, & \ldots \end{pmatrix}$$

Classical result.

$$\mu \text{ positive } \Leftrightarrow \text{ all } \gamma_i \geq 0,$$

and all $\{\beta_i\}, \{\gamma_i \geq 0\}$ occur.

Jacobi parameters.

Definition 1.

Tridiagonal matrix

$$J = \begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & \ddots \\ 1 & \beta_1 & \gamma_1 & 0 & \ddots \\ 0 & 1 & \beta_2 & \gamma_2 & \ddots \\ 0 & 0 & 1 & \beta_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}; \qquad J^n = \begin{pmatrix} m_n & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

Jacobi parameters.

Definition 2.

Cauchy transform

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x) = \frac{1}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \frac{m_3}{z^4} + \ldots.$$

Then

$$G_\mu(z) = \cfrac{1}{z - \beta_0 - \cfrac{\gamma_0}{z - \beta_1 - \cfrac{\gamma_1}{z - \beta_2 - \cfrac{\gamma_2}{z - \beta_3 - \cfrac{\gamma_3}{z - \ldots}}}}}.$$

Jacobi parameters.

Definition 3.

$\{P_n\}$ = monic orthogonal polynomials with respect to μ. Then

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x).$$

Convolution semigroups.

Measures frequently come in convolution semigroups

$$\{\mu_t\}, \qquad \mu_t * \mu_s = \mu_{t+s},$$

$$(\nu * \tau)(f) = \iint f(x+y)\, d\nu(x)\, d\tau(y).$$

Processes with independent increments.

$$J(\mu_t) = \begin{pmatrix} \beta_0(t), & \beta_1(t), & \beta_2(t), & \ldots \\ \gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \ldots \end{pmatrix}$$

In general, rational functions of t. But:

Examples.

Gaussian semigroup

$$\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}.$$

Jacobi parameters $\beta_n(t) = 0, \qquad \gamma_n(t) = (n+1)t$.

Poisson semigroup

$$\mu_t(x) = e^{-t} \sum_{k=0}^{\infty} \frac{1}{k!} t^k \delta_k(x)$$

Jacobi parameters $\beta_n(t) = t + n, \qquad \gamma_n(t) = (n+1)t$.

Linear in t!

Meixner class.

More generally: μ in Meixner class if

$$\beta_n = a + nb,$$

$$\gamma_n = (n+1)[c + nd].$$

Then (not obvious!) for μ_t,

$$\beta_n(t) = at + nb,$$

$$\gamma_n(t) = (n+1)[ct + nd].$$

Main result.

Question.

What are all convolution semigroups with Jacobi parameters polynomial in t?

Theorem.

Only Meixner. In particular, all *linear* in t.

Ingredients of the proof.

Recall

$$J^n = \begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & \ddots \\ 1 & \beta_1 & \gamma_1 & 0 & \ddots \\ 0 & 1 & \beta_2 & \gamma_2 & \ddots \\ 0 & 0 & 1 & \beta_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}^n = \begin{pmatrix} m_n & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}.$$

Formula for m_n in terms of β_i, γ_i.

Viennot-Flajolet: lattice paths.

Accardi-Bożejko: non-crossing partitions.

Moment-Jacobi formula.

$$m_n = \sum_{\pi \in NC_{1,2}(n)} \prod_{\substack{V \in \pi \\ |V|=1}} \beta_{d(V,\pi)} \prod_{\substack{V \in \pi \\ |V|=2}} \gamma_{d(V,\pi)}.$$

$NC_{1,2}$ = non-crossing partitions into pairs and singletons.
$d(V, \pi)$ = depth of V in π.

$$\begin{aligned} m_1 &= \beta_0, \\ m_2 &= \beta_0^2 + \gamma_0, \\ m_3 &= \beta_0^3 + 2\beta_0\gamma_0 + \beta_1\gamma_0, \\ m_4 &= \beta_0^4 + 3\beta_0^3\gamma_0 + 2\beta_0\beta_1\gamma_0 + \beta_1^2\gamma_0 + \gamma_0^2 + \gamma_0\gamma_1. \end{aligned}$$

Moment-cumulant formula.

$m_n(\mu_t) =$ complicated. Use cumulants instead.

$$r_1, r_2, r_3, \ldots$$

$$r_n(t) = t \cdot r_n \qquad \text{(linearize convolution).}$$

$$m_n = \sum_{\pi \in \mathcal{P}(n)} \prod_{V \in \pi} r_{|V|}.$$

No simple relation between $\{r_n\}$ and $\{\beta_n, \gamma_n\}$.

Indirect proof.

$$\beta_0(t) = at, \qquad \beta_1(t) = at + b,$$
$$\gamma_0(t) = ct, \qquad \gamma_1(t) = 2ct + d.$$

To show: $\beta_2, \gamma_2, \ldots$ the same as for Meixner(a, b, c, d).
By induction.

$$m_{2n+1}(\mu_t) = m_{2n+1}(\mu_t^M) + [\beta_n(\mu_t) - \beta_n(\mu_t^M)]\,\gamma_{n-1}(t)\ldots\gamma_1(t)\gamma_0(t).$$

$$m_k = r_k + Q_k(r_1, r_2, \ldots, r_{k-1}).$$

So

$$r_{2n+1}(\mu_t) = r_{2n+1}(\mu_t^M) + [\beta_n(\mu_t) - \beta_n(\mu_t^M)]\,\gamma_{n-1}(t)\ldots\gamma_1(t)\gamma_0(t).$$

Know $\gamma_0(t) = ct$ and $\gamma_1(t)$ depends on t, the rest of $\gamma_i(t)$ polynomial.

Indirect proof.

By induction.

$$m_{2n+1}(\mu_t) = m_{2n+1}(\mu_t^M) + [\beta_n(\mu_t) - \beta_n(\mu_t^M)]\,\gamma_{n-1}(t)\ldots\gamma_1(t)\gamma_0(t).$$

$$m_k = r_k + Q_k(r_1, r_2, \ldots, r_{k-1}).$$

So

$$r_{2n+1}(\mu_t) = r_{2n+1}(\mu_t^M) + [\beta_n(\mu_t) - \beta_n(\mu_t^M)]\,\gamma_{n-1}(t)\ldots\gamma_1(t)\gamma_0(t).$$

Know $\gamma_0(t) = ct$ and $\gamma_1(t)$ depends on t, the rest of $\gamma_i(t)$ polynomial.

$$\Rightarrow \beta_n(\mu_t) = \beta_n(\mu_t^M).$$

Use m_{2n+2} to show $\gamma_n(\mu_t) = \gamma_n(\mu_t^M)$.

Other convolutions.

Free cumulants and convolution.

$$m_n = \sum_{\pi \in NC(n)} \prod_{V \in \pi} r^{\boxplus}_{|V|}.$$

In particular

$$r^{\boxplus}_n(\mu^{\boxplus t}) = t \cdot r^{\boxplus}_n(\mu).$$

Boolean cumulants and convolution.

$$m_n = \sum_{\pi \in Int(n)} \prod_{V \in \pi} r^{\uplus}_{|V|}.$$

$$r^{\uplus}_n(\mu^{\uplus t}) = t \cdot r^{\uplus}_n(\mu).$$

Generalization.

Same proof works in the abstract (Hasebe, Saigo) setting if

1. $r_n(\mu^t) = t \cdot r_n(\mu)$.
2. $m_n = r_n + Q(r_1, \ldots, r_{n-1})$.
3. $\gamma_1(t)$ depends on t.

Theorem: *if* have a 4-parameter Meixner-type family, have no others.

Free case.

$\gamma_1(t) = ct + d$.

Answer: free Meixner class.

If for μ,

$$J(\mu) = \begin{pmatrix} a, & a+b, & a+b, & \dots \\ c, & c+d, & c+d, & \dots \end{pmatrix}$$

then for its free convolution power (not obvious!)

$$J(\mu^{\boxplus t}) = \begin{pmatrix} at, & at+b, & at+b, & \dots \\ ct, & ct+d, & ct+d, & \dots \end{pmatrix}.$$

4-parameter. The only free convolution semigroups with Jacobi parameters polynomial in t.

Boolean case.

$\gamma_1(t) = d$ independent of t. General theorem does not apply!

In fact,

$$J(\mu^{\uplus t}) = \begin{pmatrix} \beta_0 t, & \beta_1, & \beta_2, \ldots \\ \gamma_0 t, & \gamma_1, & \gamma_2, \ldots \end{pmatrix}$$

for **any** μ.

Better proof for the free case.

Recall
$$m_n = \sum_{\pi \in NC_{1,2}(n)} \prod_{\substack{V \in \pi \\ |V|=1}} \beta_{d(V,\pi)} \prod_{\substack{V \in \pi \\ |V|=2}} \gamma_{d(V,\pi)}.$$

$$m_n = \sum_{\pi \in NC(n)} \prod_{V \in \pi} r^{\boxplus}_{|V|}.$$

Both non-crossing. Młotkowksi: direct relation.

$\pi \in NC_{1,2}(n)$.

u a labeling of π:

$$u(V) \in \{0, 1, 2, \ldots, d(V, \pi)\}.$$

Some labelings connected.

$$r^{\boxplus}_n = \sum_{\substack{\text{connected} \\ \text{labelings}}} \prod_{\substack{V \in \pi \\ |V|=1}} (\beta_{u(V)} - \beta_{u(V)-1}) \prod_{\substack{V \in \pi \\ |V|=2}} (\gamma_{u(V)} - \gamma_{u(V)-1}).$$

Free cumulant-Jacobi formula.

$$
\begin{aligned}
r_1^{\boxplus} &= \beta_0, \\
r_2^{\boxplus} &= \gamma_0, \\
r_3^{\boxplus} &= \gamma_0\big[(\beta_1 - \beta_0)\big], \\
r_4^{\boxplus} &= \gamma_0\big[(\beta_1 - \beta_0)^2 + (\gamma_1 - \gamma_0)\big], \\
r_5^{\boxplus} &= \gamma_0\big[(\beta_1 - \beta_0)^3 + 3(\gamma_1 - \gamma_0)(\beta_1 - \beta_0) + (\gamma_1 - \gamma_0)(\beta_2 - \beta_1) \\
&\quad + \gamma_0(\beta_2 - \beta_1)\big], \\
r_6^{\boxplus} &= \gamma_0\big[(\beta_1 - \beta_0)^4 + 6(\gamma_1 - \gamma_0)(\beta_1 - \beta_0)^2 + 4\gamma_1(\beta_2 - \beta_1)(\beta_1 - \beta_0)^2 \\
&\quad + \gamma_1(\beta_2 - \beta_1)^2 + 2(\gamma_1 - \gamma_0)^2 + \gamma_1(\gamma_2 - \gamma_1)\big].
\end{aligned}
$$

$\beta_0(t) = at$, $\gamma_0(t) = ct$, $\beta_1(t) - \beta_0(t) = b$, $\gamma_1(t) - \gamma_0(t) = d$, $\beta_2(t) - \beta_1(t) = 0$, $\gamma_2(t) - \gamma_1(t) = 0$, etc.

Two-state convolution.

Two measures $(\widetilde{\mu}, \mu)$.

$r_n(\mu) =$ free cumulants of μ.

Two-state free cumulants $R_n(\widetilde{\mu}, \mu)$ defined via

$$m_n(\widetilde{\mu}) = \sum_{\pi \in NC(n)} \prod_{V \in \mathrm{Outer}(\pi)} R_{|V|}(\widetilde{\mu}, \mu) \prod_{U \in \mathrm{Inner}(\pi)} r_{|U|}(\mu).$$

Convolution powers:

$$r_n(\mu_t) = t \cdot r_n(\mu)$$

and

$$R_n(\widetilde{\mu}_t, \mu_t) = t \cdot R_n(\widetilde{\mu}, \mu).$$

Generalizes both free and Boolean convolution.

Two-state convolution.

Theorem.

If neither $\widetilde{\mu}$ nor μ are point masses,
Jacobi parameters of $\widetilde{\mu}_t$ polynomial in t if and only if

$$J(\widetilde{\mu}_t) = \begin{pmatrix} \widetilde{a}t, & at + \widetilde{b}, & at + b, & at + b, & \ldots \\ \widetilde{c}t, & ct + \widetilde{d}, & ct + d, & ct + d, & \ldots \end{pmatrix}$$

and

$$J(\mu_t) = \begin{pmatrix} at, & at + b, & at + b, & \ldots \\ ct, & ct + d, & ct + d, & \ldots \end{pmatrix}.$$

Note: polynomial dependence of the Jacobi parameters of μ_t automatic.

Extended Boolean case.

Proposition. $\widetilde{\mu}$ and u arbitrary.

$(\widetilde{\mu}, \delta_u)$ can be extended to a $\boxplus_c$ convolution semigroup $(\widetilde{\mu}_t, \delta_{tu})$ with linear Jacobi parameters.

Properties of $\widetilde{\mu}$.

1. Satisfy

$$d\widetilde{\mu}(x) = \frac{\sqrt{\text{quadratic}}}{\text{cubic}} + \text{ at most } 3 \text{ atoms}.$$

2. Appear in the two-state free Central and Poisson limit theorems.
3. Appear in the $q = 0$ case of the Bryc, Matysiak, Wesołowski (2007) quadratic harnesses.
4. Are precisely the measures found by Bożejko and Bryc (2009) to have a two-state free Laha-Lukacs characterization.
5. Satisfy

$$R_{n+2}(\widetilde{\mu}, \mu) = \widetilde{b}\, R_{n+1}(\widetilde{\mu}, \mu) + \widetilde{c} \sum_{k=2}^{n} R_k(\widetilde{\mu}, \mu)\, r_{n-k}(\mu).$$