K-divisible Non Crossing Partitions and Free Probability

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Outline

1. Non Crossing Partitions
    - Definitions.
    - NC(n) as a lattice.
    - K-divisible Non Crossing Partitions.
2. Free Probability
    - Basic Definitions
    - Free Cumulants
    - K-divisible elements
    - Free Convolutions
3. Further Development
    - Convolution with functions other than zeta.
    - Anular Partitions and Second Order Freeness

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Non-Crossing Partitions
Definitions

- We call $\pi = \{V_1, \ldots, V_r\}$ a partition of the set S if and only if V_i $(1 \leq i \leq r)$ are pairwise disjoint, non-void subsets of S, s.t $V_1 \cup \cdots \cup V_r = S$. We call $V_1, \cdots, V_r$ the blocks of π.
- The set of all partitions of S is denoted by$P(S)$. When $S = \{1, \ldots, n\}$ we will talk about $P(n)$.
- A partition $\pi = \{V_1, \ldots, V_r\}$ is called **crossing** if there are V_i, V_j with $j \neq i$ and $a < b < c < d$ such that $a, c \in V_i$ and $b, c \in V_j$.
- If is not crossing, then it is called **non-crossing**. The set of non-crossing partitions will be denoted by $NC(S)$. When $S = \{1, \ldots, n\}$ then we will talk about $NC(n)$.

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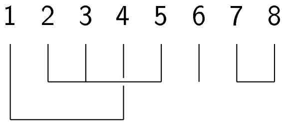
Non-Crossing Partitions
Linear Representation

Crossing Partition

$$\pi = \{\{1,4\},\{2,3,5\},\{6\},\{7,8\}\},$$



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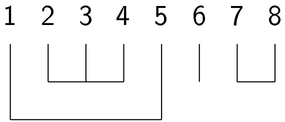

Non-Crossing Partitions
Circular Representation

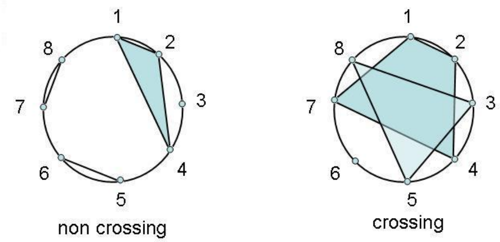

Figure: Circular Representation

Outline

Non-Crossing Partitions
NC(n) as a lattice.

We define the partial order $(P(n), \leq)$ as follows: Let π and σ be two partitions in $P(n)$, we say that $\pi \leq \sigma$ if every block $V \in \pi$ is contained completely in some block $W \in \sigma$

The partial order $(P(n), \leq)$ induces a lattice structure in $P(n)$.

We consider the partial order $(NC(n), \leq)$ seeing $NC(n)$ as a subposet in $P(n)$

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Non-Crossing Partitions
Combinatorial Convolution.

Let P be a finite partially ordered. For functions $f : P \to \mathbb{C}$ and $G : P^{(2)} \to \mathbb{C}$ the convolution $f * G : P \to \mathbb{C}$ is the function defined by

$$(f * G)(\sigma) = \sum_{\rho \in P, \rho \in P} f(\rho) G(\rho, \sigma)$$

The zeta function on P, ζ: $P^{(2)} \to \mathbb{C}$ is defined by

$$\zeta(\pi, \sigma) = \begin{cases} 1 & \pi \leq \sigma \\ 0 & \pi \nleq \sigma \end{cases}$$

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Non-Crossing Partitions
Multiplicative families

Let $(\alpha)_{n>0}$ be a sequence of complex numbers.

Define $f_n : NC(n) \to \mathbb{C}, n \geq 1$, by the following formula:

$$f_n(\pi) = \alpha_{|V_1|} \cdots \alpha_{|V_r|}$$

where $\pi = \{V_1, \dots, V_r\} \in NC(n)$.

$(f_n)_{n\geq 1}$ is called the **multiplicative family of functions** on NC determined by $(\alpha_n)_{n\geq 1}$.

We will use the notation

$$\alpha_\pi := f_n(\pi) = \alpha_{|V_1|} \cdots \alpha_{|V_r|}$$

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Non-Crossing Partitions
Dilated sequences

Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers.
We call$(a_n^{(k)})_{n\geq 1}$ the k-dilation of $(a_n)_{n\geq 1}$ to be the sequence s. t.

- $a_{nk}^{(k)} = a_n$
- $a_s^{(k)} = 0$ if k does not divide s.

Quick example: If $(a_n)_{n\geq 1} = \{1, 2, 3, 4, 5, 6, ...\}$ then:
$(a_n^{(2)})_{n\geq 1} = \{0, 1, 0, 2, 0, 3, 0, 4, 0, 5...\}$
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Non-Crossing Partitions
Multiplicative families

What is the relation between

$$f * \zeta * \cdots * \zeta$$

and

$$f^{(k)} * \zeta$$

when f belongs to a multiplicative family on NC ?

Non-Crossing Partitions
Main Result

Theorem (A.)

The following statements are equivalent:

1)The family $(g_n)_{n\geq 1}$ is the result of applying k times the zeta function to $(f_n)_{n\geq 1}$.($g = f * \zeta * \cdots * \zeta$)

2)The family $(g_n^{(k)})_{n\geq 1}$ is the result of applying the zeta function to $(f_n^{(k)})_{n\geq 1}$. ($g^{(k)} = f^{(k)} * \zeta$).

Remark

This property is not true for $P(n)$ or $I(n)$!!!

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Non-Crossing Partitions
k-divisibles NC.

Example

Let $\{a_n\}_{n>0} = \{1, 0, \ldots\}$, $b = a * \zeta$. *and* $c = b * \zeta * \zeta$. *Then*

$$b_n = \sum_{\pi \in NC(n)} a_\pi = a_{||\cdots|} = 1 \quad c_n = \sum_{\pi \in NC(n)} b_\pi = \sum_{\pi \in NC(n)} 1 = \#NC(n)$$

$(a_n^{(2)})_{n>0} = \{0, 1, 0, \ldots\}$ *and let* $d = a^{(2)} * \zeta$. *Then*

$$d_{2n} = \sum_{\pi \in NC(2n)} a_\pi^{(2)} = \sum_{\pi \in NC_2(2n)} 1 = \#\{\pi \in NC(2n) \mid \pi \text{ pair}\}$$

By the last theorem $d_n = c_n^{(2)}$ *or* $d_{2n} = c_n$.

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Non-Crossing Partitions

k-divisible Non Crossing Partititios

- A partition $\pi = \{V_1, \ldots, V_r\}$ is called k-equal if $V_i = k$ for all i.
- A partition $\pi = \{V_1, \ldots, V_r\}$ is called k-divisible if $k \mid V_i$ for all i.
- We will denote the set of k-divisible non crossing partitions of kn elements by $NC^{(k)}(n)$.
- A k-multichain of a POSET P is a sequence $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_k$, with $x_i \in P$.

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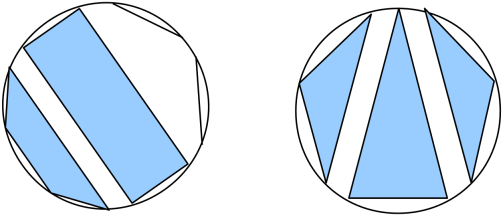

2-divisible partition 3-equal partition

Non-Crossing Partitions

Counting...

Theorem (Kreweras, 1972)

The number of $k+1$-equal partitions of $\{1,2,\ldots,(k+1)n\}$ is the Fuss-Catalan number

$$C_n^k = \frac{\binom{n(k+1)}{n}}{nk+1}$$

Non-Crossing Partitions

Counting...

Theorem

The number of k-divisible partitions of $\{1, 2, ..., kn\}$ *is the Fuss-Catalan number*

$$C_n^k = \frac{\binom{n(k+1)}{n}}{nk+1}$$

Non-Crossing Partitions

Counting....

Theorem (Edelman 1980)

The number of k-multichains of $NC(n)$ is the Fuss-Catalan number

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Non-Crossing Partitions
Counting results.

Corollary

The $k+1$-equal partitions of $[(k+1)n]$ are in bijection with the k-divisible paritítions of nk and the k-multichains of $NC(n)$.

Can we recover this from the main theorem?

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Non-Crossing Partitions

NC^(k) as a lattice.

Consider the partial order $(NC^{(k)}(n), \leq)$ seeing $NC^{(k)}(n)$ as a subposet in $NC(n)$

$(NC^{(k)}(n), \leq)$ is not a lattice. However it is a *POSET* and then we can consider a combinatorial convolution in $(NC^{(k)}(n), \leq)$.

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NC^k as a lattice.

Remark

The convolution in $NC^{(k)}$of multiplicative families is equivalent to convolution in NC with dilated sequences.

$$\beta_n = \sum_{\pi \in NC^{(k)}(n)} \alpha_\pi \iff \beta_n^{(k)} = \sum_{\pi \in NC(n)} \alpha_\pi^{(k)}$$

Non-Crossing Partitions

k multichains

Remark

The number of k-multichains with minimal element x_1 and maximal element x_k in a POSET P is given by

$$\zeta_P * \cdots * \zeta_P(x_1, x_k) = \sum_{x_1 \leq \cdots \leq x_k} 1$$

Non-Crossing Partitions
as a lattice.

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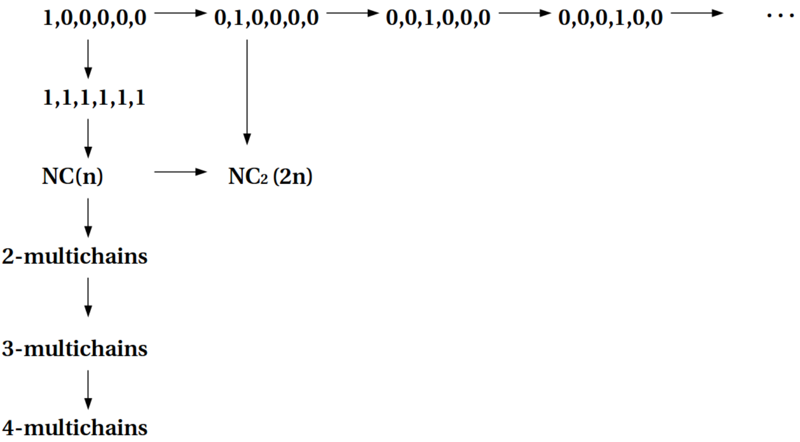
Non-Crossing Partitions

Counting...

$$
\begin{tikzcd}
1,0,0,0,0,0 \arrow[r] \arrow[d] & 0,1,0,0,0,0 \arrow[r] & 0,0,1,0,0,0 \arrow[r] & 0,0,0,1,0,0 \arrow[r] & \cdots \\
1,1,1,1,1,1 \arrow[d] & & & & \\
NC(n) \arrow[d] & & & & \\
2\text{-multichains} \arrow[d] & & & & \\
3\text{-multichains} \arrow[d] & & & & \\
4\text{-multichains} & & & &
\end{tikzcd}
$$

Non-Crossing Partitions
Counting...



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\text{NC(n)} \arrow[d] & & & & \\
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NC(n) \arrow[d] & & & & \\
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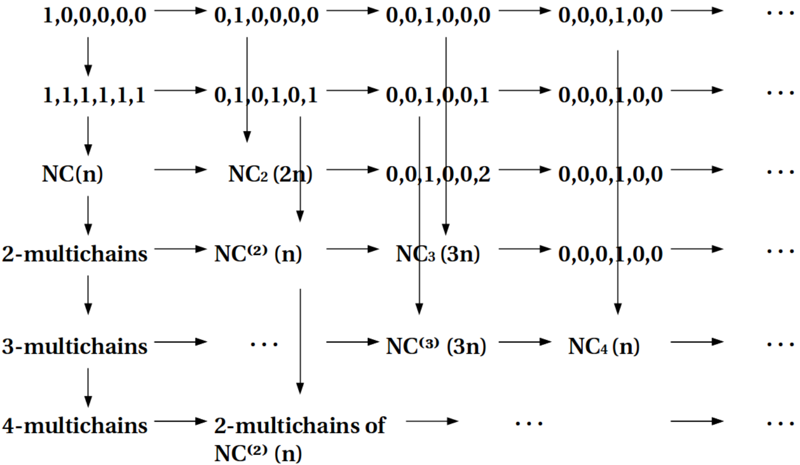
$$
\begin{tikzcd}
1,0,0,0,0,0 \arrow[r] \arrow[d] & 0,1,0,0,0,0 \arrow[r] & 0,0,1,0,0,0 \arrow[r] & 0,0,0,1,0,0 \arrow[r] \arrow[dddd] & \cdots \\
1,1,1,1,1,1 \arrow[r] \arrow[d] & 0,1,0,1,0,1 \arrow[r] & 0,0,1,0,0,1 \arrow[ddd] & & \\
NC(n) \arrow[d] & & & & \\
\text{2-multichains} \arrow[d] & & & & \\
\text{3-multichains} \arrow[r] \arrow[d] & \cdots \arrow[r] & NC^{(3)}(3n) \arrow[r] & NC_4(n) \arrow[r] & \cdots \\
\text{4-multichains} & & & &
\end{tikzcd}
$$

Non-Crossing Partitions

Counting...



Non-Crossing Partitions

Summary

Corollary

The $k+1$-equal partitions of $[(k+1)n]$ are in bijection with the k-divisible paritions of nk and the k-multichains of $NC(n)$.

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We also recover the result of Armstrong (2007) that $\#(NC^{(k)}(n))^{(l)} = \#(NC^{(kl)}(n))$.

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Outline

Free Probability
Non-Commutative Probability Spaces

A non-commutative probability is a pair $(\mathscr{A},\varphi)$, where$\mathscr{A}$ is a unital algebra and $\varphi:\mathscr{A}\to\mathbb{C}$ is a linear functional s.t. $\varphi(1)=1$.
When $\mathscr{A}$ is a C^*-algebra φ is positive we call $\mathscr{A}$ a C^*-probability space. In this frame we will talk about:

(non-commutative) random variables: $a\in\mathscr{A}$

normal elements: $a\in\mathscr{A}$ s. t. $a^*a=aa^*$

self-adjoint:$a\in\mathscr{A}$ s. t. $a=a^*$

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Free Probability
Free Independence

The unital sub-algebras $(\mathscr{A}_i)_{i\in I}$ of $\mathscr{A}$ are called **freely independent** if for any $k \in \mathbb{N}$, we have

$$\varphi(a_1, \ldots, a_k) = 0,$$

whenever:

i) $a_i \in \mathscr{A}_{j(i)}, j(1) \neq j(2) \neq \cdots \neq j(k)$ and

ii)$\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_k) = 0.$

Random variables $(a_i)_{i\in I}$ are **free** if the unital algebras generated by them are free.

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Outline

1. Non Crossing Partitions
   - Definitions.
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2. Free Probability
   - Basic Definitions
   - Free Cumulants
   - K-divisible elements
   - Free Convolutions
3. Further Development
   - Convolution with functions other than zeta.
   - Anular Partitions and Second Order Freeness

Free Probability
Free Cumulants

The **free cumulants** $(\kappa_\pi)_{\pi \in NC}$ is the multiplicative family of functionals in $NC(n)$ defined inductively by the moments in $(\mathscr{A}, \varphi)$ via the **moment-cumulant formula:** for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathscr{A}$,

$$\begin{aligned} \varphi(a_1 \ldots a_n) &= \sum_{\pi \in NC(n)} \kappa_\pi [a_1, \ldots, a_n], \\ \kappa_\pi [a_1, \ldots, a_n] &:= \prod_{V_i \in \pi} \kappa_{\mathbf{1}_{|V_i|}} \left[a_{i_1}, \ldots, a_{i_{|V_i|}} \right] \end{aligned}$$

Free Probability
Free Cumulants

In particular, for a random variable a, the free cumulants $(\kappa_\pi(a))_{\pi \in NC}$, are defined via

$$\varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi(a)$$

Equivalently if $m_n = \varphi(a^n)$ then $m = \kappa * \zeta$.

Free Probability
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In particular, for a random variable a, the free cumulants $(\kappa_\pi(a))_{\pi \in NC}$, are defined via

$$\varphi(a^n) = \sum_{\pi \in NC(n)} \kappa_\pi(a)$$

Equivalently if $m_n = \varphi(a^n)$ then $m = \kappa * \zeta$.

Free Probability

Freeness and Free Cumulants

Theorem (Vanishing of mixed Cumulants)

The non-commutative random variables$(a_i)_{i \in I}$ in $\mathscr{A}$ are freely independent iff for all $n \geq 2$, $i(1), \ldots, i(n) \in I$

$$\kappa_n \left(a_{i(1)}, \ldots, a_{i(n)}\right) = 0,$$

whenever there exist $1 \leq k, l \leq n$ with $i(k) \neq i(l)$.

Corollary. If a and b are free, then free cumulants are additive, namely

$$\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b,$$

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K-divisible elements

Defintions

An element $x \in A$ is called k-**divisible** if the only non vanishing moments are multiples of k. That is $\varphi(x^n) = 0$ whenever k does not divide s.

Let $x \in A$ be k-divisible and let $\alpha_n := \kappa_{kn}(x, \ldots, x)$. We call $(\alpha_n)_{n \geq 1}$ the k-determining sequence of x.

$x \in A$ is k-divisible if and only if the only non-vanishing free cumulants are multiples of k.

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K-divisible elements

Motivation

Theorem (Speicher, Nica)

Let (A, φ) be a non-commutative probability space and let $x \in A$ be an ***even element*** *with determining sequence $(\alpha_n)_{n\geq 1}$. Then the cumulants of x^2 are given as follows:*

$$\kappa\left(x^2, \ldots x^2\right) = \sum_{\pi \in NC(n)} \alpha_\pi.$$

K-divisible elements

Consequences

Theorem (A.)

Let (A, φ) be a non-commutative probability space and let x a k-divisible elements with k-determining sequence $(\alpha_n)_{n\geq 1}$. Then the following formula holds for the cumulants of x^k

$$\kappa_n(x^k, x^k, \ldots, x^k) = \alpha_n * \zeta_n * \cdots * \zeta_n$$

K-divisible elements

Consequences

Corollary

Let (A, φ) be a non-commutative probability space and let $x \in A$ be a k-divisible element with k-determining sequence $(\alpha_n)_{n\geq 1}$. Then the cumulants of x^k are given as follows:

$$\kappa_n\left(x^k, \ldots x^k\right) = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}.$$

K-divisible elements

Consequences

Corollary

Let (A, φ) be a non-commutative probability space and let $x \in A$ be a $k+1$-divisible element with $k+1$-determining sequence $(\alpha_n)_{n=1}^{\infty}$. Then the following formula holds for the cumulants of x^{k+1}.

$$\kappa_n\left(x^{k+1}, \ldots x^{k+1}\right) = \sum_{\pi \in NC(n)} \beta_\pi,$$

where

$$\beta_n = \sum_{\pi \in NC((k-1)n)} \alpha_\pi^{(k-1)}.$$

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Free Probability
*-distributions

Since we are interested in the $*$-**moments** of an element $a \in \mathscr{A}$ that is

$$\varphi(a^{m_1}(a^*)^{n_1}\dots(a^*)^{n_k})$$

If $a \in \mathscr{A}$ is normal and μ_a is a probability measure on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} z^l \bar{z}^k \mathrm{d}\mu(z) = \varphi\left(a^l (a^*)^k\right),$$

we call μ_a the $*$-**distribution** of a.

If $\mathscr{A}$ is a C^*-algebra and $a \in \mathscr{A}$ is normal then the $*$-distribution of a exists and is unique. If a is selfadjoint then the support of μ_a is real.

Free Probability
Free Multiplicative Convolution

Ley a be positive and b selfadjoint. If a and b are free selfadjoint with $*$-distributions μ andν, respectively, and then the **free multiplicative convolution** ofμ and ν is the distribution $a^{1/2}ba^{1/2}$ and is denoted by $\mu \boxtimes \nu$.

The operation $\boxtimes$ is associative and commutative if we restrict to positive measures.

Free Probability
Free Compound Poisson Distributions

Let $\lambda > 0$ and ν be a prob. m. supp. on $\mathbb{R}$, If μ is a prob. m. such that, for $n \geq 1$

$$\kappa_n^{(\mu)} = \lambda m_n(\nu),$$

$\pi_\lambda(\nu) := \mu$ is called the **free compound Poisson distribution** of rate λ and jump distribution ν.
We denote by $\pi_\lambda := \pi_\lambda(\delta_1)$ and by $\pi := \pi_1$.

If μ is a free compound Poisson with rate 1 and jump distribution ν then

$$\mu = \pi_1(\nu) = \pi \boxtimes \nu$$

Free Probability

Free Compound Poissons and Free Infinite Divisibility

Corollary

If x is even with disitribution μ a symmetric compound poisson with rate 1 and jump distribution ν ($\mu = \pi(1, \nu)$). Then the distribution μ^2 of x^2 is a compound poisson with rate 1 and jump distribution $\rho = \pi(1, \nu^2)$. Equivalently,

$$(\pi \boxtimes \nu)^2 = \pi^{\boxtimes 2} \boxtimes \nu^2$$

Corollary

If μ is symmetric and free infinitely divisible, then μ^2 is also free infinitely divisible.

Free Convolution
Free Compound Poissons

Corollary. Suppose that x is an k-divisible element with k-determining sequence $(\alpha_n)_{n\geq 1}$, and suppose that $(\alpha_n)_{n\geq 1}$ is a cumulant sequence of a positive element $(k_n(a) = \alpha_n)$ with distribution ρ, then

$$\mu = \pi^{\boxtimes k-1} \boxtimes \rho$$

In particular when $\alpha_n = m_n(\nu)$ then

$$\mu = \pi^{\boxtimes k} \boxtimes \nu.$$

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Further Development

Theorem (A., Perez-Abreu, 2009)

If σ is a symmetric measure and ρ is a measure with positive support then

$$(\rho \boxtimes \nu)^2 = \rho^{\boxtimes 2} \boxtimes \nu^2$$

Further Development
Convolution

Question. For which families$h$ is true that the following statements are equivalent?

1)The family $(g_n)_{n\geq 1}$ is the $k-$fold convolution of$(h_n)_{n\geq 1}$ to $(f_n)_{n\geq 1}.(g = f * h * \cdots * h\ )$

2)The family $(g_n^{(k)})_{n\geq 1}$ is the $k-$fold convolution of$(h_n)_{n\geq 1}$ to $(f_n^{(k)})_{n\geq 1}$. $(g^{(k)} = f^{(k)} * h)$.

Outline

Further Development

Counting results.

The following is a bijection between the $k+1$-equal partitions of $kn+n$ and the k-divisible partitions of nk elements.

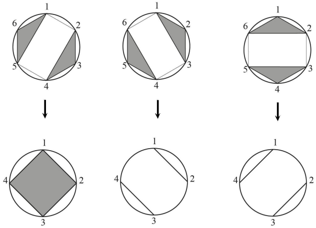

Figure: Bijection between for k=2

Further Development
Anular Partitions

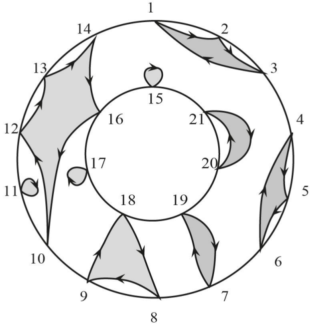

Figure: Anular Partition

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Thanks

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