Matricial R-transform

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  - classical independence
  - linearization formula for the logarithm of the Fourier transform

$$\log F_{\mu_1 * \mu_2} = \log F_{\mu_1} + \log F_{\mu_2}$$

- Free probability
  - free independence
  - linearization formula for the R-transform of Voiculescu

$$R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$$

Noncommutative independence

There are different notions of noncommutative independence:

- axiomatic theory
  - freeness (Voiculescu)
  - boolean independence (Bożejko, Speicher, Woroudi)
  - monotone independence (Muraki)
- generalizations
  - conditional freeness (Bożejko, Speicher, Leinert)
  - conditionally monotone independence (Hasebe)
- related to subordination
  - freeness with subordination (R.L.)
  - orthogonal independence (R.L.)

Convolutions and transforms

Additive convolutions

- with a linearization formula for the associated transform
  - free $\mu_1 \boxplus \mu_2$
  - boolean $\mu_1 \uplus \mu_2$
  - c-free $(\mu_1, \nu_1) \boxplus (\mu_2, \nu_2)$
- with no linearization formula for the associated transform
  - monotone $\mu_1 \rhd \mu_2$
  - s-free $\mu_1 \boxvdash \mu_2$
  - orthogonal $\mu_1 \vdash \mu_2$

Matricial freeness and strong matricial freeness

We propose two closely related notions of independence

- *matricial freeness*
- *strong matricial freeness*

Strong matricial freeness

- leads to *unification* of noncommutative independence
- is related to *subordination* in free probability

Matricial freeness

- is related to *random matrices*

Array of subalgebras

Array of subalgebras

Let $(\mathcal{A}, \varphi)$ be a *-noncommutative probability space and let $(\mathcal{A}_{i,j})$ be a two-dimensional array of *-subalgebras of $\mathcal{A}$ such that

1. each $\mathcal{A}_{i,j}$ has an *internal unit* $1_{i,j}$ which is a projection and for which it holds that
$$1_{i,j}a_{i,j} = a_{i,j}1_{i,j} = a_{i,j}$$
for any $a_{i,j} \in \mathcal{A}_{i,j}$
2. the unital algebra generated by all internal units, called the *algebra of units* and denoted $\mathcal{I}$, is commutative.

Array of states

Array of states

Let φ be a distinguished state on $\mathcal{A}$. The state φ_j is called a *conjugate state* if it is defined in terms of φ as

$$\varphi_j(a) = \varphi(b_j^* a b_j)$$

where $b_j \in \mathcal{A}_{j,j} \cap \mathrm{Ker}\varphi$. We assume from now on that we have a two-dimensional array of subalgebras and states of the form

$$\begin{pmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \varphi_{2,1} & \varphi_{2,2} \end{pmatrix} = \begin{pmatrix} \varphi & \varphi_2 \\ \varphi_1 & \varphi \end{pmatrix}$$

i.e. the diagonal states agree with φ and the off-diagonal states are conjugate states.

Sets of indices

Sets of indices

Let us introduce subsets of $(\{1,2\} \times \{1,2\})^m$ of the form

$$\Gamma_m \quad = \quad \{((i_1, i_2), (i_2, i_3), \ldots, (i_m, i_{m+1})) : i_1 \neq i_2 \neq \ldots \neq i_m\}$$

where $m \in \mathbb{N}$ and let

$$\Gamma = \bigcup_{m=1}^{\infty} \Gamma_m$$

be the corresponding union.

Strongly matricially free array of units

Strongly matricially free array of units

We say that $(1_{i,j})$ is a *strongly matricially free array of units* associated with $(\mathcal{A}_{i,j})$ and $(\varphi_{i,j})$ if for any diagonal state φ it holds that

1. $\varphi(u_1 a u_2) = \varphi(u_1)\varphi(a)\varphi(u_2)$ for any $a \in \mathcal{A}$ and $u_1, u_2 \in \mathcal{I}$,
2. $\varphi(1_{i,j}) = \delta_{i,j}$ for any i, j,
3. if $a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k}$, where $1 < k \leqslant m$, then

$$\varphi(a1_{i_1,j_1}a_2 \ldots a_m) = \begin{cases} \varphi(aa_2 \ldots a_n) & ((i_1,j_1),\ldots,(i_m,j_m)) \in \Gamma \\ 0 & \text{otherwise} \end{cases}.$$

where $a \in \mathcal{A}$ is arbitrary and $(i_1, j_1) \neq \ldots \neq (i_m, j_m)$.

Strong matricial freeness

Strong matricial freeness

We say that *-subalgebras $(\mathcal{A}_{i,j})$ are *strongly matricially free* with respect to $(\varphi_{i,j})$ if

1. the array $(1_{i,j})$ is a strongly matricially free array of units
2. it holds that

$$\varphi(a_1 a_2 \ldots a_n) = 0 \quad \text{whenever} \quad a_k \in \mathcal{A}_{i_k,j_k} \cap \mathrm{Ker}\varphi_{i_k,j_k},$$

where $(i_1, j_1) \neq \ldots \neq (i_n, j_n)$.

Shape of array determines independence

Shape of array determines independence

Under suitable assumptions on considered states, strong matricial freeness gives a correspondence between different shapes of matrices and different types of independence

- square arrays $\rightarrow$ freeness
- lower-triangular arrays $\rightarrow$ monotone independence
- diagonal arrays $\rightarrow$ boolean independence
- anti-upper-triangular arrays $\rightarrow$ freeness with subordination
- one-column arrays $\rightarrow$ orthogonal independence

Strongly matricially free Fock space

Strongly matricially free Fock space

By the *strongly matricially free Fock space* over the array of Hilbert spaces $(\mathcal{H}_{i,j})$ we understand the Hilbert space direct sum

$$\mathcal{N} = \mathbb{C}\Omega \oplus \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \ldots \neq i_m \\ n_1, \ldots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{i_m,i_m}^{\otimes n_m}$$

where Ω is a unit vector, with the canonical inner product.

Properties:

- *freeness* : neighboring indices are different
- *matriciality* : neighboring pairs are matricially related
- *diagonal subordination* : last pair is diagonal

Summands

If $(\mathcal{H}_{i,j})$ is a two-dimensional square array and consists of one-dimensional Hilbert spaces $\mathcal{H}_{i,j} = \mathbb{C}e_{i,j}$, the first few summands are of the form

$$
\begin{aligned}
\mathcal{N}^{(0)} &= \mathbb{C}\Omega \\
\mathcal{N}^{(1)} &= \mathbb{C}e_{1,1} \oplus \mathbb{C}e_{2,2} \\
\mathcal{N}^{(2)} &= \mathbb{C}e_{1,1}^{\otimes 2} \oplus \mathbb{C}e_{2,2}^{\otimes 2} \oplus \mathbb{C}(e_{1,2} \otimes e_{2,2}) \oplus \mathbb{C}(e_{2,1} \otimes e_{1,1}) \\
\mathcal{N}^{(3)} &= \mathbb{C}e_{1,1}^{\otimes 3} \oplus \mathbb{C}e_{2,2}^{\otimes 3} \oplus \mathbb{C}(e_{2,1} \otimes e_{1,1}^{\otimes 2}) \oplus \mathbb{C}(e_{1,2} \otimes e_{2,2}^{\otimes 2}) \\
&\oplus \mathbb{C}(e_{2,1}^{\otimes 2} \otimes e_{1,1}) \oplus \mathbb{C}(e_{1,2}^{\otimes 2} \otimes e_{2,2}) \\
&\oplus \mathbb{C}(e_{1,2} \otimes e_{2,1} \otimes e_{1,1}) \oplus \mathbb{C}(e_{2,1} \otimes e_{1,2} \otimes e_{2,2}),
\end{aligned}
$$

etc.

Creation and annihilation operators

Creation and annihilation operators

Let $\mathcal{N}$ be the strongly matricially free Fock space over the array $(\mathcal{H}_{i,j}) = (\mathbb{C}e_{i,j})$ and let

$$\tau : \mathcal{N} \to \mathcal{F}(\bigoplus_{i,j} \mathcal{H}_{i,j})$$

be the associated embedding. By the *strongly matricially free creation operators* we understand operators of the form

$$\ell_{i,j} = \alpha_{i,j}\tau^{*}\ell(e_{i,j})\tau$$

where $\alpha_{i,j} > 0$, and the *strongly matricially free annihilation operators* are their adjoints.

Action of creation operators

Non-trivial action of the creation operators

$$
\begin{aligned}
\ell_{1,1}\Omega &= \alpha_{1,1} e_{1,1} \\
\ell_{2,2}\Omega &= \alpha_{2,2} e_{2,2} \\
\ell_{1,1} e_{1,1}^{\otimes n} &= \alpha_{1,1} e_{1,1}^{\otimes (n+1)} \\
\ell_{2,2} e_{2,2}^{\otimes n} &= \alpha_{2,2} e_{2,2}^{\otimes (n+1)} \\
\ell_{1,2} e_{2,2}^{\otimes n} &= \alpha_{1,2} (e_{1,2} \otimes e_{2,2}^{\otimes n}) \\
\ell_{2,1} e_{1,1}^{\otimes n} &= \alpha_{2,1} (e_{2,1} \otimes e_{1,1}^{\otimes n}) \\
\ell_{2,1} (e_{2,1}^{\otimes k} \otimes e_{1,1}^{\otimes n}) &= \alpha_{2,1} (e_{2,1}^{\otimes (k+1)} \otimes e_{1,1}^{\otimes n}) \\
\ell_{1,2} (e_{1,2}^{\otimes k} \otimes e_{2,2}^{\otimes n}) &= \alpha_{1,2} (e_{1,2}^{\otimes (k+1)} \otimes e_{2,2}^{\otimes n})
\end{aligned}
$$

Canonical example

Strongly matricially free array of *-algebras

If $\mathcal{A}_{i,j} = alg(\ell_{i,j}, \ell_{i,j}^{*})$, where $\ell_{i,j}^{*}\ell_{i,j} = \alpha_{i,j}^{2}1_{i,j}$, then the array $(\mathcal{A}_{i,j})$ is strongly matricially free with respect to $(\varphi_{i,j})$, where the diagonal states agree with the vacuum states and the off-diagonal states are conjugate states φ_j defined by vectors $e_{j,j}$, where $j \in \{1,2\}$.

Remark

If we denote $\mathcal{F}_{j,j} = \mathcal{F}(\mathbb{C}e_{j,j})$, then

1. the unit $1_{j,j}$ is the projection onto $\mathcal{F}_{j,j}$ for $j \in \{1,2\}$
2. the unit $1_{1,2}$ is the projection onto $\mathcal{N} \ominus \mathcal{F}_{1,1}$
3. the unit $1_{2,1}$ is the projection onto $\mathcal{N} \ominus \mathcal{F}_{2,2}$

Strongly matricially free convolution

Strongly matricially free convolution

Let $(a_{i,j})$ be a two-dimensional array of strongly matricially free random variables with the corresponding array of distributions $(\mu_{i,j})$ in the states $(\varphi_{i,j})$. The φ-distribution of the sum

$$A = \sum_{i,j} a_{i,j} \quad \text{denoted} \quad \boxplus_{i,j} \mu_{i,j}$$

will be called the *strongly matricially free convolution* of $(\mu_{i,j})$.

Addition of rows gives binary convolutions

Addition of rows gives binary convolutions

If the variables are row-identically distributed, the strongly matricially free convolution gives the following binary convolutions

- if the array is square, then $\boxplus_{i,j}\mu_{i,j} = \mu_1 \boxplus \mu_2$
- if the array is lower-triangular, then $\boxplus_{i,j}\mu_{i,j} = \mu_1 \rhd \mu_2$
- if the array is diagonal, then $\boxplus_{i,j}\mu_{i,j} = \mu_1 \uplus \mu_2$
- if the array is upper-anti-triangular, then $\boxplus_{i,j}\mu_{i,j} = \mu_1 \boxbar \mu_2$
- if the array is a column, then $\boxplus_{i,j}\mu_{i,j} = \mu_1 \vdash \mu_2$

Toeplitz operators in free probability

Toeplitz operators in free probability

Voiculescu used Toeplitz operators to prove the linearization formula for the R-transform. A new proof was given by Haagerup who used the adjoints

$$a = \ell_1 + f(\ell_1^*) \quad \text{and} \quad b = \ell_2 + g(\ell_2^*)$$

where ℓ_1, ℓ_2 are free creation operators on the full Fock space $\mathcal{F}(\mathcal{H})$ over a two-dimensional Hilbert space with orthonormal basis $\{e_1, e_2\}$ and where f, g are polynomials.

Strongly matricially free Toeplitz operators

Strongly matricially free Toeplitz operators

Let $(\ell_{i,j})$ be the array of strongly matricially free creation operators on $\mathcal{N}$ and let $f_{i,j}$ be a polynomial for any $(i,j) \in J$. Operators of the form

$$a_{i,j} = \ell_{i,j} + f_{i,j}(\ell_{i,j}^{*}) \tag{4.2}$$

where $(i,j) \in J$ and the constant term of $f_{i,j}$ is the internal unit $1_{i,j}$ multiplied by a complex number, will be called *strongly matricially free Toeplitz operators* .

Vacuum state and conjugate states

Vacuum state and conjugate states

We will need the distributions of Toeplitz operators in the array of states $(\varphi_{i,j})$ defined by unit vectors $(\Omega_{i,j})$, where

$$\Omega_{j,j} = \Omega \quad \text{and} \quad \Omega_{i,j} = e_{j,j} \;\; \text{for} \;\; i \neq j,$$

which replace the single vacuum vector in the free case.

Proposition

The R-transform of the distribution $\mu_{i,j}$ of the operator $a_{i,j}$ in the state $\varphi_{i,j}$ is given by

$$R_{i,j}(z) = f_{i,j}(\alpha_{i,j}^2 z),$$

where $(i,j) \in J$ and the constant term of $f_{i,j}$ is a complex number.

Lemma 1

Lemma 1

Consider the vector

$$\rho(z) = (1 - zL)^{-1}\Omega, \quad \text{where} \quad L = \sum_{i,j} \ell_{i,j}$$

where $|z| < (\sum_{i,j} |\alpha_{i,j}|^2)^{-1}$. The sum $A = \sum_{i,j} a_{i,j}$ of strongly matricially free Toeplitz operators satisfies the equation

$$A\rho(z) = \frac{1}{z}(\rho(z) - \Omega) + \sum_{i,j} f_{i,j}(\alpha_{i,j}^2 z) 1_{i,j} \rho(z)$$

where $0 < |z| < (\sum_{i,j} |\alpha_{i,j}|^2)^{-1}$.

Lemma 2

Lemma 2

Let φ be the state associated with the vacuum vector Ω. Then there exists ϵ such that

$$z = \varphi\left(\left(\frac{1}{z} + \sum_{i,j} R_{i,j}(z) 1_{i,j} - A\right)^{-1}\right)$$

whenever $0 < |z| < \epsilon$.

Noncommutative distribution of $a \in \mathcal{A}$

Definition

The collection of mixed moments of the form

$$\varphi(b_{n_1} a b_{n_2} \ldots b_{n_{m-1}} a b_{n_m}), \text{where } b_{n_k} \in \mathcal{I} \text{ for } 1 \leqslant k \leqslant m \text{ and } m \in \mathbb{N}$$

will be called the *distribution of a* in the state φ.

Analog of the Cauchy transform

Let $\mathcal{A}$ be a Banach algebra with a subalgebra $\mathcal{I}$ and let $b \in \mathcal{I}$ be invertible with $\| b^{-1} \| < \| a \|^{-1}$ then the inverse of $b - a$ exists and takes the form

$$(b-a)^{-1} = \sum_{n=0}^{\infty} b^{-1}(ab^{-1})^n$$

which converges in the norm topology. This leads to the operatorial analog of the Cauchy transform of the form

$$\mathcal{G}_a(b) = \sum_{n=0}^{\infty} \varphi\left(b^{-1}(ab^{-1})^n\right),$$

due to continuity of φ, which plays the role of the Cauchy transform of the b-distribution of A in the state φ.

Operatorial R-transform

Operatorial R-transform

Let μ denote the distribution of $a \in \mathcal{A}$ in the state φ. If there exists an $\mathcal{I}$-valued power series of the form

$$\mathcal{R}_a(z) = \sum_{n=1}^{\infty} c_n z^{n-1},$$

where $c_n \in \mathcal{I}$ for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, which is convergent in the norm topology for sufficiently small $|z|$, and for which it holds that

$$\mathcal{G}_a \left(\frac{1}{z} + \mathcal{R}_a(z) \right) = z$$

whenever $|z|$ is sufficiently small and positive, it will be called an *operatorial R-transform of the distribution μ* .

Necessary and sufficient conditions

Lemma

An $\mathcal{I}$-valued power series $\mathcal{R}(z) = \sum_{n=1}^{\infty} c_n z^{n-1}$ converging in the norm topology in a neighborhood of zero is an operatorial R-transform of the φ-distribution of A if and only if

$$\sum_{k=1}^{m} \sum_{n_1+\ldots+n_k=m-k} \varphi(b_{n_1} A b_{n_2} \ldots b_{n_{k-1}} A b_{n_k}) = 0$$

for all $m \geqslant 2$, where we assume that $n_1, \ldots, n_k$ are non-negative integers and where the series $B(z) = \sum_{n=0}^{\infty} b_n z^{n+1}$ is the multiplicative inverse of $C(z) = 1/z + \mathcal{R}(z)$.

Matricial R-transform

Matricial R-transform

Let $A \in \mathcal{A}$ be the sum of random variables $(a_{i,j})$ in a unital complex C^*-algebra $\mathcal{A}$ which are strongly matricially free with respect to $(\varphi_{i,j})$ and let $\mathcal{I}$ be its unital C^*-subalgebra generated by the internal units. If an $\mathcal{I}$-valued operatorial R-transform $\mathcal{R}_A$ of the φ-distribution of A takes the form

$$\mathcal{R}_A(z) = \sum_{i,j} \mathcal{R}_{i,j}(z) = \sum_{i,j} R_{i,j}(z) 1_{i,j}$$

where $R_{i,j}$ is the R-transform of $\mu_{i,j}$, the distribution of $a_{i,j}$ in the state $\varphi_{i,j}$, it will be called a *matricial R-transform* of the noncommutative distribution of A in φ.

Existence theorem

Existence theorem

If $(a_{i,j})$ is an array of random variables from a unital complex C^*-algebra $\mathcal{A}$ which is strongly matricially free with respect to $(\varphi_{i,j})$ and $(R_{i,j})$ is the corresponding array of R-transforms, then

$$\mathcal{R}_A(z) = \sum_{i,j} \mathcal{R}_{i,j}(z),$$

where $A = \sum_{i,j} a_{i,j}$ and $\mathcal{R}_{i,j}(z) = R_{i,j}(z)1_{i,j}$ for any $(i,j) \in J$, with sufficiently small $|z|$, is an operatorial R-transform of the distribution of A in φ.

Special cases

Free

If the array is square and row-identically distributed, then the matricial R-transform associated with $\mathcal{G}_A$ takes the form

$$\mathcal{R}_A(z) = R_{\mu_1}(z)1_{\mathcal{A}} + R_{\mu_2}(z)1_{\mathcal{A}}$$

and can be identfied with the scalar-valued R-transform of $\mu_1 \boxplus \mu_2$.

Special cases

Boolean

If the array is diagonal, then the matricial R-transform associated with $\mathcal{G}_A$ takes the form

$$\mathcal{R}_A(z) = R_{\mu_1}(z)1_{1,1} + R_{\mu_2}(z)1_{2,2}$$

which linearizes the extended boolean convolution.

Special cases

Monotone

If the array is lower-triangular and row-identically distributed, then the matricial R-transform associated with $\mathcal{G}_A$ takes the form

$$\mathcal{R}_A(z) = R_{\mu_1}(z)1_{1,1} + R_{\mu_2}(z)1_{\mathcal{A}}$$

which linearizes the extended monotone convolution.

Special cases

c-free

In the general case we can write the matricial R-transform associated with $\mathcal{G}_A$ as

$$\mathcal{R}_A(z) = \sum_{i,j} \mathcal{Q}_{i,j}(z),$$

where $\mathcal{Q}_{i,j}(z) = Q_{i,j}(z)q_{i,j}$ for any i,j and $(q_{i,j})$ is an array of orthogonal projections defined in terms of $(1_{i,j})$, with $(Q_{i,j})$ being the array of R-transforms of free convolutions

$$\left(\begin{array}{cc} \mu_1 \boxplus \mu_2 & \mu_1 \boxplus \nu_2 \\ \mu_2 \boxplus \nu_1 & \nu_1 \boxplus \nu_2 \end{array} \right),$$

where $\mu_{1,1} = \mu_1$, $\mu_{2,2} = \mu_2$, $\mu_{1,2} = \nu_1$ and $\mu_{2,1} = \nu_2$.

Projections P_j

Projections P_j

Introduce canonical projections

$$P_j : \mathcal{N} \to \mathcal{N}(j), \quad \text{where } j \in \{1,2\},$$

where

$$\mathcal{N}(j) = \bigoplus_{m=1}^{\infty} \bigoplus_{\substack{i_1 \neq \ldots \neq i_{m-1} \neq j \\ n_1, \ldots, n_m \in \mathbb{N}}} \mathcal{H}_{i_1,i_2}^{\otimes n_1} \otimes \mathcal{H}_{i_2,i_3}^{\otimes n_2} \otimes \ldots \otimes \mathcal{H}_{j,j}^{\otimes n_m},$$

and $\mathcal{H}_{i,k} = \mathbb{C}e_{i,k}$ for any i, k.

Lemma

Lemma

Let $\mathcal{R}_A$ be the matricial R-transform and let $A_j = P_j A P_j$ for $j \in \{1, 2\}$. Then

$$\mathcal{G}_{A_j}\left(\frac{1}{z} + \mathcal{R}_{A_j}(z)\right) = z$$

for small $|z| > 0$, where $\mathcal{G}_{A_j}$ is the Cauchy transform associated with the distribution of A_j in the conjugate state φ_j and $\mathcal{R}_{A_j} = \mathcal{R}_A P_j$.

Matricial formulation

We can write the results for all states in the array form. For that purpose, we introduce the array of transforms and the associated variables, namely

$$(\mathcal{G}_{i,j}) = \begin{pmatrix} \mathcal{G}_A & \mathcal{G}_{A_2} \\ \mathcal{G}_{A_1} & \mathcal{G}_A \end{pmatrix} \quad \text{and} \quad (A_{i,j}) = \begin{pmatrix} A & A_2 \\ A_1 & A \end{pmatrix},$$

and we consider their distributions in the array of states $(\varphi_{i,j})$.

Matricial formulation

Matricial formulation

With the above notations, it holds that

$$\mathcal{G}_{i,j}\left(\frac{1}{z}+\mathcal{R}_A(z)\right)=z$$

for small $|z|>0$ and $i,j\in\{1,2\}$.

Uniqueness theorem

Uniqueness theorem

There exists a unique operatorial R-transform associated with the array $(\mathcal{G}_{i,j})$ of the form

$$\mathcal{R}_A(z) = \sum_{i,j} \mathcal{T}_{i,j}(z)$$

where $\mathcal{T}_{i,j}(z) = T_{i,j}(z)1_{i,j}$ for any $(i,j) \in J$ and each $T_{i,j}(z)$ is a power series converging in some neighborhood of zero

Partitioned colored cumulants

Consider the coloring σ of blocks of $\pi \in \mathcal{NC}_m$ by numbers from the set $\{1, 2\}$ which leads to a colored partition (π, σ) with blocks

$$B(\pi, \sigma) = \{(\pi_1, \sigma), (\pi_2, \sigma), \ldots, (\pi_r, \sigma)\}.$$

Then assign to each block (π_k, σ)

1. the free cumulant $r_{i,j}$ if π_k is colored by i and its nearest outer block is colored by j
2. the free cumulant $r_{j,j}$ if π_k is a covering block colored by j

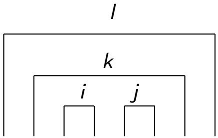
Writing $r(\pi_k, \sigma) = r_{i,j}(n_k)$, where n_k is the cardinality of π_k, define

$$r[\pi, \sigma] = r(\pi_1, \sigma) r(\pi_2, \sigma) \ldots r(\pi_r, \sigma),$$

called the *partitioned colored cumulant* associated with (π, σ).

Example of a partitioned cumulant

First, label each block with a number from the set $\{1,2\}$.
Then compute the partitioned colored cumulant as follows.



$$\rightarrow \qquad r[\pi,\sigma] = r_{i,k} r_{j,k} r_{k,l} r_{l,l}$$

Lemma

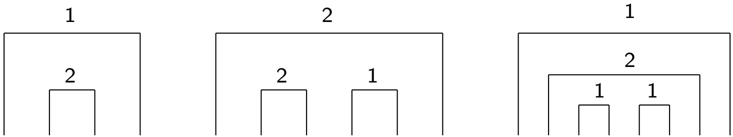
Lemma

Let A be the sum of strongly matricially free random variables $(a_{i,j})$. If μ is the φ–distribution of the sum of A and $\mu_{i,j}$ is the $\varphi_{i,j}$–distribution of $a_{i,j}$, then

$$M_{\mu}(m) = \sum_{(\pi,\sigma)\in\mathcal{NC}_m^{c}} r_{\mu}[\pi,\sigma]$$

where the summation extends over all admissible colorings, i.e. compatible with the strongly matricially free product.

Partitioned colored cumulants



$$r[\pi,\sigma] = r_{1,1}r_{2,1} \qquad r[\chi,\sigma'] = r_{2,2}^2 r_{1,2} \qquad r[\zeta,\sigma''] = r_{1,1}r_{2,1}r_{1,2}^2$$

All addmissible colorings

All adimissible colorings

If we collect all admissible colorings for the considered partitions, we obtain the sums of partitioned colored cumulants over all admissible colorings:

$$
\begin{aligned}
r[\pi] &= r_{1,1}(r_{1,1}+r_{2,1})+r_{2,2}(r_{2,2}+r_{1,2}),\\
r[\chi] &= r_{1,1}(r_{1,1}+r_{2,1})^2+r_{2,2}(r_{2,2}+r_{1,2})^2\\
r[\zeta] &= r_{1,1}^2(r_{1,1}+r_{2,1})^2+r_{1,1}r_{2,1}r_{1,2}^2\\
&\quad +r_{2,2}^2(r_{2,2}+r_{1,2})^2+r_{2,2}r_{1,2}r_{2,1}^2
\end{aligned}
$$

Free case

If we set $r_{1,2} = r_{1,1} = r_1$ and $r_{2,1} = r_{2,2} = r_2$ (row-identically distributed square array), we obtain

$$\begin{aligned} r[\pi] &= r_1^2 + 2r_1r_2 + r_2^2, \\ r[\chi] &= r_1^3 + 3r_1^2r_2 + 3r_1r_2^2 + r_2^3, \\ r[\zeta] &= r_1^4 + 3r_1^3r_2 + 2r_1^2r_2^2 + 3r_1r_2^3 + r_2^4, \end{aligned}$$

which is the contribution from partitions π, χ and ζ to the moments of $\mu_1 \boxplus \mu_2$.

Monotone case

Monotone case

If we set $r_{2,1} = r_{2,2} = r_2$, $r_{1,1} = r_1$ and $r_{1,2} = 0$ (row-identically distributed lower-triangular array), we obtain

$$\begin{aligned} r[\pi] &= r_1^2 + r_1 r_2 + r_2^2, \\ r[\chi] &= r_1^3 + r_1^2 r_2 + r_1 r_2^2 + r_2^3, \\ r[\zeta] &= r_1^4 + 2r_1^3 r_2 + r_1^2 r_2^2 + r_2^4, \end{aligned}$$

which gives the contribution from π, χ and ζ to the moments of $\mu_1 \rhd \mu_2$.

Moments

Moments

The lowest order moments of A are expressed in terms of free cumulants of the measures $\mu_{i,j}$ as follows:

$$\begin{aligned} M_\mu(1) &= r_{1,1}(1) + r_{2,2}(1), \\ M_\mu(2) &= r_{1,1}(2) + r_{2,2}(2) + (r_{1,1}(1) + r_{2,2}(1))^2, \\ M_\mu(3) &= r_{1,1}(3) + r_{2,2}(3) + 2(r_{1,1}(2) + r_{2,2}(2))(r_{1,1}(1) + r_{2,2}(1)) \\ &+ r_{1,1}(2)(r_{1,1}(1) + r_{2,1}(1)) + r_{2,2}(2)(r_{2,2}(1) + r_{1,2}(1)) \\ &+ (r_{1,1}(1) + r_{2,2}(1))^3. \end{aligned}$$

If the array is square and row-identically distributed, these moments agree with the moments of $\mu_1 \boxplus \mu_2$.
If that array is lower-triangular and row-identically distributed, these moments agree with those of $\mu_1 \rhd \mu_2$.