Kerov character polynomials: recent progress in asymptotic representation theory of symmetric groups

(joint work with Maciej Dołęga and Valentin Féray)

Piotr Śniady

University of Wroclaw

Outlook

- What can we say about the asymptotics of characters of symmetric groups $S(n)$ in the limit $n \to \infty$?
- Exact values of characters can be calculated from free cumulants thanks to Kerov polynomials.
- The main result: explicit combinatorial interpretation of the coefficients of Kerov polynomials.
- Open problems: relations to Schubert calculus, Toda hierarchy, ...

Plan

- Representations of symmetric groups
  - Representations
  - Young diagrams and normalized characters
  - Free cumulants
- Kerov character polynomials
- Open problems
- Proof of Kerov conjecture

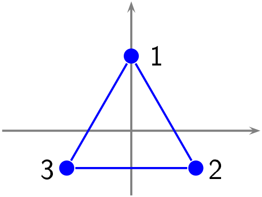
Representations

representation of a group G is a homomorphism from G to invertible $n \times n$ matrices

$$\rho : G \to M_{n\times n}(\mathbb{C}).$$

Example

Representation of $S(3)$ as symmetries of a triangle on a plane.



Irreducible representations

A representation $\rho : G \to \mathrm{End}(V)$ on a vector space V is reducible if there exists a nontrivial decomposition into subrepresentations.

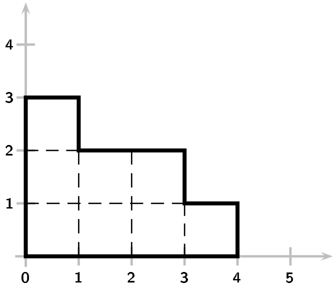
Otherwise, a representation is called irreducible.

Motivations:

- irreducible representations $\longleftrightarrow$ Fourier transform,
- harmonic analysis on groups,
- random walks on groups,
- ...

Irreducible representations of symmetric groups

Irreducible representations ρ^λ of symmetric group $S(n)$ are indexed by Young diagrams λ having n boxes.



Very combinatorial object, not good for asymptotic problems.

Dilations of diagrams

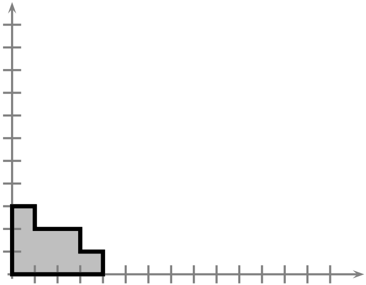
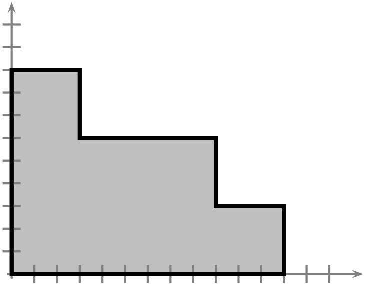


diagram λ



dilated diagram $s\lambda$ for $s = 3$

Normalized characters

For $\pi \in S(k)$ and irreducible representation ρ^λ of $S(n)$ (assume $k \leq n$) we define the normalized character

$$\Sigma_\pi^\lambda = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\operatorname{Tr} \rho^\lambda(\pi)}{\text{dimension of } \rho^\lambda}.$$

Most interesting case: characters on cycles

$$\Sigma_k^\lambda = \Sigma_{(1,2,\dots,k)}^\lambda.$$

Problem

For fixed $k \geq 1$ what can we say about $\Sigma_k^{s\lambda}$ for $s \to \infty$?

Free cumulants

The map $s \mapsto \Sigma_{k-1}^{s\lambda}$ is a polynomial of degree k.
We define free cumulants $R_2^\lambda, R_3^\lambda, \dots$ of diagram λ to be asymptotically the dominant terms of the character on cycles:

$$R_k^\lambda = \lim_{s\to\infty} \frac{1}{s^k} \Sigma_{k-1}^{s\lambda} = [s^k] \Sigma_{k-1}^{s\lambda}.$$

Advertisement

Free cumulants are very very nice quantities to describe a Young diagram: they can be explicitly calculated in several approaches and are very useful in asymptotic representation theory.

Free cumulants are homogeneous with respect to dilations: $R_k^{s\lambda} = s^k R_k^\lambda$.

Plan

- Representations of symmetric groups
- Kerov character polynomials
  - Kerov polynomials
  - Combinatorics of Kerov polynomials
  - Applications of the main result
- Open problems
- Proof of Kerov conjecture

Kerov polynomials

Free cumulants give approximations of characters:

$$\Sigma_k \approx R_{k+1},$$

but they can also give exact values of characters thanks to Kerov character polynomials:

$$\begin{aligned}
\Sigma_1 &= R_2, \\
\Sigma_2 &= R_3, \\
\Sigma_3 &= R_4 + R_2, \\
\Sigma_4 &= R_5 + 5R_3, \\
\Sigma_5 &= R_6 + 15R_4 + 5R_2^2 + 8R_2, \\
\Sigma_6 &= R_7 + 35R_5 + 35R_3R_2 + 84R_3.
\end{aligned}$$

Kerov conjecture

Theorem/Conjecture (Kerov)

For each $k \geq 1$ there exists a universal polynomial $K_k(R_2, R_3, \dots)$ with non-negative integer coefficients called Kerov character polynomial such that

$$\Sigma_k = K_k(R_2, R_3, \dots)$$

What is the combinatorial interpretation of coefficients?

Féray: Kerov's conjecture is true, coefficients have a complicated combinatorial interpretation.

Linear terms of Kerov polynomials

For a permutation π we denote by $C(\pi)$ the set of cycles of π.

Theorem (Biane and Stanley)

The coefficient $[R_\ell]K_k$ *is equal to the number of pairs* (σ_1, σ_2) *where*

- $\sigma_1, \sigma_2 \in S(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \ldots, k)$,
- $|C(\sigma_2)| = 1$,
- $|C(\sigma_1)| + |C(\sigma_2)| = \ell$.

Quadratic terms of Kerov polynomials

For a permutation π we denote by $C(\pi)$ the set of cycles of π.

Theorem (Féray)

The coefficient $[R_{\ell_1} R_{\ell_2}]K_k$ is equal to the number of triples (σ_1, σ_2, q) with the following properties:

- $\sigma_1, \sigma_2 \in S(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \ldots, k)$,
- $|C(\sigma_2)| = 2$,
- $|C(\sigma_1)| + |C(\sigma_2)| = \ell_1 + \ell_2$,
- $q : C(\sigma_2) \to \{\ell_1, \ell_2\}$ *is a surjective map on cycles of* σ_2;
- *for each cycle c of σ_2 there are more than $q(c) - 1$ cycles of σ_1 which intersect nontrivially c.*

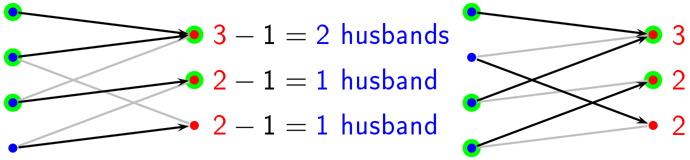
The main result: combinatorial interpretation of Kerov polynomials

Theorem (Dołęga, Féray, Śniady)

The coefficient $[R_2^{s_2} R_3^{s_3} \cdots] K_k$ *is equal to the number of triples* (σ_1, σ_2, q) *such that*

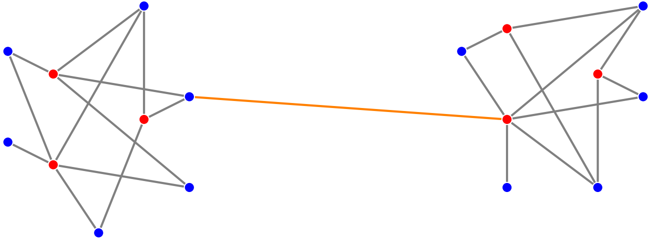
- $\sigma_1, \sigma_2 \in S(k)$ *are such that* $\sigma_1 \circ \sigma_2 = (1, 2, \ldots, k)$,
- $|C(\sigma_2)| = s_2 + s_3 + \cdots$,
- $|C(\sigma_1)| + |C(\sigma_2)| = 2s_2 + 3s_3 + 4s_4 + \cdots$,
- $q : C(\sigma_2) \to \{2, 3, \ldots\}$ *is a coloring such that each color* $i \in \{2, 3, \ldots\}$ *is used* s_i *times,*
- *for every nontrivial set* $\emptyset \subsetneq A \subsetneq C(\sigma_2)$ *of cycles of* σ_2 *there are more than* $\sum_{c \in A} \left(q(c) - 1\right)$ *cycles of* σ_1 *which intersect* $\bigcup A$.

Marriage interpretation



Example: coefficient $[R_2^2 R_3]K_k$. For given σ_1, σ_2 we consider a bipartite graph $\mathcal{V}_{\sigma_1,\sigma_2}$ with the vertices corresponding to cycles of σ_1 (boys) and cycles of σ_2 (girls). We draw an edge if two cycles intersect (boy is allowed to marry a girl). Each boy wants to marry one girl and each girl $g \in C(\sigma_2)$ wants to marry $q(g) - 1$ boys. We require that it is possible to arrange marriages and that for each non-trivial set of girls the set of their husbands is not uniquely determined.

Restriction on graphs



Corollary

If there exists an disconnecting edge with at least one girl in both components then the factorization cannot contribute (no matter which labeling we choose).

Application: coefficients of Kerov polynomials are small.

Applications of the main result

- positivity: Kerov polynomials give characters as simple sums without too many cancellations,
- optimal estimates for characters,
- more information on the structure of Kerov polynomials (Lassalle's conjectures)

Plan

Representations of symmetric groups

Kerov character polynomials

Open problems

- Exotic interpretations of Kerov polynomials
- Open problems

Proof of Kerov conjecture

Exotic interpretations of Kerov polynomials

Conjecture

Maybe coefficients of Kerov polynomials

- *are equal to dimensions of some intersection (co)homologies of Schubert varieties? [conjecture of Philippe Biane]*
- *are equal to something related to moduli space of analytic maps on Riemann surfaces? or ramified coverings of a sphere? [conjecture of Śniady]*
- *are algebraic solutions to some integrable hierarchy (Toda?) and their coefficients are related to the tau function of the hierarchy? [conjecture of Jonathan Novak]*

Open problems

- free cumulants originally come from Voiculescu's free probability theory / random matrix theory...
  is there some analogue of Kerov character polynomials in the random matrix theory / respresentation theory of the unitary groups $U(d)$?
- is it possible to study Kerov polynomials in such a scaling that phenomena of universality of random matrices occur?
- the structure of Kerov polynomials is still not clear (Goulden–Rattan conjecture, Lassalle's conjectures)

Conjecture: C-expansion of characters

Subdominant term of the character:

$$C^{\lambda}_{k-1} = \lim_{s\to\infty} \frac{1}{s^{k-1}} \left(\Sigma^{s\lambda}_{k} - R^{s\lambda}_{k+1} \right) = [s^{k-1}] \left(\Sigma^{s\lambda}_{k} - R^{s\lambda}_{k+1} \right)$$

Conjecture (Goulden and Rattan)

For each $k \geq 1$ there exists a universal polynomial L_k called Goulden–Rattan polynomial with rational (non-negative?) coefficients (with relatively small denominators?) such that

$$\Sigma_k - R_{k+1} = L_k(C_2, C_3, \dots).$$

Plan

Representations of symmetric groups

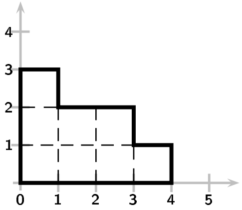
Kerov character polynomials

Open problems

Proof of Kerov conjecture

- Fundamental functionals $S_2, S_3, \ldots$ of shape
- Stanley polynomials
- Toy example: quadratic terms of Kerov polynomials

Fundamental functionals $S_2, S_3, \ldots$ of shape



$$\text{contents}_{(x,y)} = x - y$$

Fundamental functionals of shape of λ:

$$S_n^\lambda = (n-1) \iint_{(x,y)\in\lambda} (\text{contents}_{(x,y)})^{n-2}\, dx\, dy$$

- easy to compute,
- homogeneous: $S_n^{s\lambda} = s^n S_n^\lambda$,
- there are explicit formulas which express functionals $S_2, S_3, \ldots$ in terms of free cumulants $R_2, R_3, \ldots$ and conversely... therefore free cumulants can be explicitly calculated from the shape of a Young diagram!

Relation between functionals $S_2, S_3, \ldots$ and free cumulants $R_2, R_3, \ldots$

$$S_n = \sum_{l \geq 1} \frac{1}{l!} (n-1)_{l-1} \sum_{\substack{k_1, \ldots, k_l \geq 2 \\ k_1 + \cdots + k_l = n}} R_{k_1} \cdots R_{k_l},$$

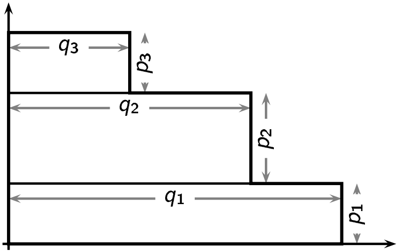
$$R_n = \sum_{l \geq 1} \frac{1}{l!} (-n+1)^{l-1} \sum_{\substack{k_1, \ldots, k_l \geq 2 \\ k_1 + \cdots + k_l = n}} S_{k_1} \cdots S_{k_l},$$

Example:

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \mathcal{F} + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1 + k_2}} \mathcal{F}.$$

All derivatives at $R_2 = R_3 = \cdots = S_2 = S_3 = \cdots = 0$.

Stanley polynomials



For numbers $p_1, p_2, \dots, q_1, q_2, \dots$ we consider multirectangular (generalized) Young diagram $\mathbf{p} \times \mathbf{q}$.

Theorem (conjectured by Stanley, proved by Féray)

For any permutation π the normalized character $\Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$ is a polynomial in $p_1, p_2, \dots, q_1, q_2, \dots$, called Stanley polynomial, for which there is an explicit formula.

Stanley-Féray character formula

Theorem (conjectured by Stanley, proved by Féray)

For $\pi \in S(n)$

$$\Sigma_{\pi}^{\mathbf{p}\times\mathbf{q}} = \sum_{\substack{\sigma_1,\sigma_2\in S(n)\\ \sigma_1\circ\sigma_2=\pi}} \sum_{\phi_2:C(\sigma_2)\to\mathbb{N}} (-1)^{\sigma_1} \cdot \prod_{b\in C(\sigma_1)} q_{\phi_1(b)} \cdot \prod_{c\in C(\sigma_2)} p_{\phi_2(c)},$$

where coloring $\phi_1 : C(\sigma_1) \to \mathbb{N}$ *is defined by*

$$\phi_1(c) = \max_{\substack{b\in C(\sigma_2),\\ b \text{ and } c \text{ intersect}}} \phi_2(b) \qquad \text{for } c \in C(\sigma_1)$$

The Stanley polynomial depends on the graph $\mathcal{V}_{\sigma_1,\sigma_2}$.

Stanley-Féray character formula, toy version

Corollary

For $\pi \in S(n)$

$$(-1)[p_1 q_1^i p_2 q_2^j]\Sigma_\pi^{\mathbf{p}\times\mathbf{q}}$$

is equal to the number of factorizations $\pi = \sigma_1 \circ \sigma_2$ such that

- *σ_1 has $i + j$ cycles,*
- *$\sigma_2 = \{c_1, c_2\}$ has two (labeled) cycles,*
- *there are exactly j cycles of σ_1 which intersect c_2.*

The Stanley polynomial depends on the graph $\mathcal{V}_{\sigma_1,\sigma_2}$.

Stanley polynomials and functionals $S_2, S_3, \dots$

Theorem

If $\mathcal{F}$ is a sufficienly nice function on the set of generalized Young diagrams then it as a polynomial in $S_2, S_3, \dots$.

$$\frac{\partial}{\partial S_{k_1}} \cdots \frac{\partial}{\partial S_{k_l}} \mathcal{F} \bigg|_{S_2 = S_3 = \cdots = 0} = [p_1 q_1^{k_1 - 1} \cdots p_l q_l^{k_l - 1}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}}$$

- Therefore expansion of Σ_π in terms of $S_2, S_3, \dots$ can be extracted from Stanley polynomials.
- Stanley polynomials are explicitly given by Stanley-Féray formula and depend on geometry of bipartite graphs $\mathcal{V}_{\sigma_1, \sigma_2}$.
- Once we know the expansion of Σ_π in terms of $S_2, S_3, \dots$ we can find expansion of Σ_π in terms of free cumulants $R_2, R_3, \dots$.

Free cumulants vs fundamental functionals

Free cumulants $R_2, R_3, \dots$

- describe Young diagram in language of representation theory
- best quantities for calculating characters

Functionals $S_2, S_3, \dots$

- describe Young diagram in language of its shape
- directly related to Stanley polynomials

Toy example: $[R_{k_1} R_{k_2}]\Sigma_n$

$$\frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \mathcal{F} = \frac{\partial^2}{\partial S_{k_1} \partial S_{k_2}} \mathcal{F} + (k_1 + k_2 - 1) \frac{\partial}{\partial S_{k_1 + k_2}} \mathcal{F} =$$

$$[p_1 p_2 q_1^{k_1 - 1} q_2^{k_2 - 1}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}} + (k_1 + k_2 - 1)[p_1 q_1^{k_1 + k_2 - 1}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1 - 1} q_2^{k_2 - 1}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}} - [p_1 p_2 q_2^{k_1 + k_2 - 2}] \mathcal{F}^{\mathbf{p} \times \mathbf{q}}$$

Toy example: $[R_{k_1} R_{k_2}]\Sigma_n$

We are interested in factorizations $\sigma_1 \circ \sigma_2 = (1, \ldots, n)$ such that σ_1 has $k_1 + k_2 - 2$ cycles and $\sigma_2 = \{c_1, c_2\}$ has two cycles.

#(fact. such that c_1 has $\geq k_1$ friends, c_2 has $\geq k_2$ friends) =

#(all fact.) − #(fact. such that c_1 has $\leq k_1 - 1$ friends)

−#(fact. such that c_2 has $\leq k_2 - 1$ friends) =

$$(-1) \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j}} \left[p_1 p_2 q_1^i q_2^j\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}} + \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq i\leq k_1-1}} \left[p_1 p_2 q_1^j q_2^i\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}}$$

$$+ \sum_{\substack{i+j=k_1+k_2-2,\\ 1\leq j\leq k_2-1}} \left[p_1 p_2 q_1^i q_2^j\right] \Sigma_k^{\mathbf{p}\times\mathbf{q}} =$$

$$[p_1 p_2 q_1^{k_1-1} q_2^{k_2-1}]\Sigma_n^{\mathbf{p}\times\mathbf{q}} - [p_1 p_2 q_2^{k_1+k_2-2}]\Sigma_n^{\mathbf{p}\times\mathbf{q}} = \frac{\partial^2}{\partial R_{k_1} \partial R_{k_2}} \Sigma_n$$

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