The Normal Law, Free Probability and a Hopf Algebra of Rooted Binary Trees

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Contents

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Noncommutative Probability Spaces

Definition

A *noncommutative probability space* is a pair $(\mathcal{A}, \varphi)$ where

- $\mathcal{A}$ is a unital algebra over $\mathbf{C}$.
- $\varphi : \mathcal{A} \to \mathbf{C}$ is a unital linear functional.
- Elements $X \in \mathcal{A}$ are called *noncommutative random variables*.
- The *distribution* of a random variable is the linear functional μ_X on polynomials formally written as $\varphi(X^k) = \int t^k d\mu_X(t)$.
- If $\mathcal{A}$ is a C^*-algebra, φ is a state and X is selfadjoint, then its distribution is a probability measure on the real line.

Classical vs Free Independence

Classical Independence

A family of subalgebras $\mathcal{A}_i \subseteq \mathcal{A}$ is *independent* if

$$\forall i_1, i_2, \dots i_n \text{ distinct}, \forall X_j \in \mathcal{A}_{i_j}$$

we have $\varphi(X_1 \cdots X_n) = \varphi(X_1)\,\varphi(X_2) \cdots \varphi(X_n)$

Free Independence

A family of subalgebras $\mathcal{A}_i \subseteq \mathcal{A}$ is *free* if $\varphi(X_1 \cdots X_n) = 0$ whenever $X_j \in \mathcal{A}_{i_j}$, $\varphi(X_j) = 0$, $i_j \neq i_{j+1}$.

In both cases the joint distribution of an independent resp. free family is uniquely determined by the individual distributions.

Classical vs Free Convolution

Classical	Free
$X,\ Y$ independent $\Downarrow$ $\mu_{X+Y} = \mu_X * \mu_Y$	$X,\ Y$ free $\Downarrow$ $\mu_{X+Y} = \mu_X \boxplus \mu_Y$
μ infinitely divisible $\Updownarrow$ $\forall n\, \exists \mu_n : \mu = \mu_n * \cdots * \mu_n$	μ infinitely divisible $\Updownarrow$ $\forall n\, \exists \mu_n : \mu = \mu_n \boxplus \cdots \boxplus \mu_n$

Examples	
Normal distribution	Wigner semicircle law
Poisson distribution	Marchenko-Pastur distribution
Cauchy distribution	Cauchy distribution
Dirac distribution	Dirac distribution

These are examples of the **Bercovici-Pata bijection**.

Classical $\cap$ Free Infinite Divisibility

There are a few measures which are both classically and freely infinitely divisible:

1. Dirac distribution δ_x
2. Cauchy distribution $d\mu(x) = \frac{dx}{\pi(1+x^2)}$
3. $1/2$-free stable law (coincides with β-distribution of 2nd kind), $d\beta(x) = \frac{(4x-1)^{1/2}}{x^2}\,dx$.

Theorem B-B-L-S 2008

The normal distribution is also of this kind.

What we have: analytic proof (Nevanlinna theory)
What we want: combinatorial proof

Criteria for infinite divisibility

Classical Lévy-Khinchin formula

μ is classically infinitely divisible $\iff$ the *cumulant function* $K_\mu(\zeta) = \log \mathbf{E} e^{i\zeta X} = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} (i\zeta)^n$ has a representation

$$K_\mu(\zeta) = i\gamma\zeta + \int \left(e^{i\zeta t} - 1 - \frac{i\zeta t}{1+t^2}\right) \frac{1+t^2}{t^2} \, d\rho(x)$$

Free Lévy-Khinchin formula

μ is $\boxplus$-infinitely divisible $\iff$ the *Voiculescu transform* $\varphi_\mu(z) = \sum_{n=1}^{\infty} c_n z^{-n-1}$ has a representation

$$\varphi_\mu(z) = \gamma + \int \frac{1+tz}{z-t} \, d\rho(t)$$

The coefficients κ_n / c_n are called the classical / free *cumulants*.

More criteria for infinite divisibility

Analytic criterion

A probability measure μ on the real line is $\boxplus$-infinitely divisible

$$\Longleftrightarrow$$

Voiculescu transform extends to an analytic function $\mathbf{C}^+ \to \mathbf{C}^-$.

Can be used in our case!

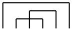
Combinatorial criterion

A probability measure μ is classically/freely infinitely divisible $\Longleftrightarrow$ the shifted sequence of classical/free cumulants $(\kappa_{n+2})_{n\geq 1}$ (resp. $(c_{n+2})_{n\geq 1}$) is positive definite, i.e., can be interpreted as the sequence of moments of some measure.

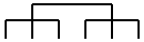
So far we were not able to exploit the combinatorial criterion, although there are many combinatorial interpretations (see below).

Set partitions

Denote Π_n the lattice of partitions of the set $\{1, 2, \ldots, n\}$. These partitions can be depicted as diagrams, e.g.,



non-connected



connected

A partition is called *connected* if the corresponding diagram is a connected graph. Notation: Π_n^{conn}.

Lemma

The free cumulants can be expressed in terms of classical cumulants

$$c_n = \sum_{\pi \in \Pi_n^{\mathrm{conn}}} \kappa_\pi$$

Here as usual $\kappa_\pi = \prod_{B \in \pi} \kappa_{|B|}$.

Matchings

Corollary

The free cumulants of the normal distribution $N(0,1)$ are given by

$$c_n = \begin{cases} s_m & \text{if } n = 2m \text{ even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Here $s_m = \left|\Pi_{2m}^{(2,\mathrm{conn})}\right|$ denotes the number of *pair partitions* or *matchings*, i.e., partitions into blocks of size 2.

$$1, 1, 4, 27, 248, \ldots$$

These numbers have been considered by Touchard, Riordan, Stein, Wilf, Flajolet, ...
Broadhurst/Kreimer: *"rainbow approximation for anomalous dimensions of Yukawa theory at spacetime dimension $d = 4$"*.

A recursion

Proposition (Riordan)

The numbers s_m satisfy the recursion

$$s_n = (n-1)\sum_{i=1}^{n-1} s_i s_{n-i}$$

We are interested in the shifted sequence

$$s'_n := s_{n+1} \qquad \Longrightarrow \qquad s'_n = n\sum_{i=0}^{n-1} s'_i s'_{n-1-i}$$

Compare with Catalan numbers: $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$.

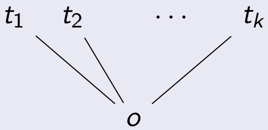
Yet the Jacobi parameters are not integers.

Trees: factorials

And now for something completely different.

Definition

Let t be a finite rooted tree. It can be decomposed into the root o and rooted subtrees $t_1, t_2, \ldots, t_k$:

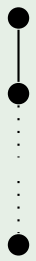


We write $t = t_1 \vee t_2 \vee \cdots \vee t_k$ for the *grafting* of the tree. Define the *tree factorial* recursively by

$$t! = |t| \cdot t_1! \cdot t_2! \cdots t_k!$$

Trees: factorials

Example



$$! = n!$$

Definition

Denote $\mathcal{PRBT}_n$ the set of planar rooted binary trees with n nodes.

Proposition

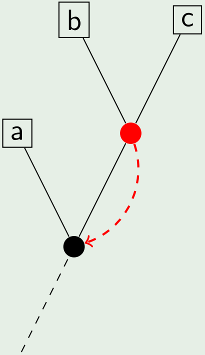
$$s'_n = \sum_{t \in \mathcal{PRBT}_n} t!$$

Proof: Same recursion is satisfied.

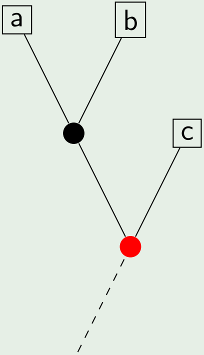
Trees: Markov chains

What is $t!$?

1. For general trees, $n!/t!$ is the number of ways to grow the tree t by successively adding vertices.
2. For $t \in \mathcal{PRBT}_n$, $\pi(t) = 1/t!$ is the stationary distribution of the *move-to-root* Markov chain (Dobrow/Fill).



$\longrightarrow$



Trees: Markov chains

It is also the stationary distribution of the *Naimi-Trehel algorithm* from computer science.

Corollary

$$\sum_{t \in \mathcal{PRBT}_n} \frac{1}{t!} = 1$$

By a well-known fact from Markov chain theory we have

Corollary

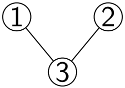
$$s'_n = \sum_{t \in \mathcal{PRBT}_n} \mathbf{E}_t T_t = C_n \mathbf{E} T$$

where $\mathbf{E}_t T_t$ is the expected time of first return of the random walk starting at t and $\mathbf{E} T$ is the expected time of first return to a randomly chosen starting tree t.

Trees: Orderings

Let t be a planar rooted binary tree on n vertices.
A labeling with integers 1, 2,. . . , n is called *LR-monotone* or *anti-increasing* if

- the labels are distinct
- for every vertex v, the labels on the left subtree are smaller than the labels on the right subtree.



Equivalently, every antichain in the tree has increasing labels.
This is *not* the same as binary search trees.

Proposition

For $t \in \mathcal{PRBT}$ the number of LR-monotone labelings equals $t!$.

Trees: Loday-Ronco Hopf algebra

Last remarks for those who know Hopf algebras.

Product of trees

Let $s = s_1 \vee s_2$, $t = t_1 \vee t_2 \in \mathcal{PRBT}$. Define the product

$$s * t = s_1 \vee (s_2 * t) + (s * t_1) \vee t_2$$

Coproduct

$$\Delta(t) = t \otimes | + \sum_u u_1 * u_2 * \cdots * u_k \otimes u$$

where the sum runs over all nonempty subtrees u of t with the same root and u_1, $u_2, \ldots, u_k$ are the components of the forest $t \setminus u$.

- Formal linear span $\mathbf{C}\mathcal{PRBT}$ becomes a graded Hopf algebra.
- Homogeneous components have dimensions C_n (Catalan numbers).

Trees: Monotone Loday-Ronco Hopf algebra

Proposition

The Loday-Ronco operations are compatible with LR-monotone orderings. Thus $\mathbf{C}\mathcal{LRPRBT}$ becomes a graded Hopf algebra whose homogeneous components have dimensions s_n'.

Corollary

The shifted Voiculescu transform of the normal law is the Hilbert series of the this Hopf algebra.

Q: Are there criteria when the dimension sequence of a graded Hopf algebra is positive definite?