# The free log-normal distribution and confluent hypergeometric series

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**Definition of the free multiplicative analog measures**

Hari Bercovici and Dan Voiculesu. *Lévy-Hinčin type theorems for multiplicative and additive free convolution*, Pacific J. Math., 1992.

## Voiculescu's $S$-transform

Let $\mu$ a compact supported probability measure on $\mathbb{C}$ with $\mathrm{m}_1(\mu) \neq 0$.

- $\Psi_\mu(z) := \int_{\mathbb{C}} \frac{zt}{1-zt} \, d\mu(t) = \sum_{n=1}^{\infty} \mathrm{m}_\mathrm{n}(\mu) z^n$

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- $S_\mu(z) := \frac{z+1}{z}\, \chi_\mu(z)$

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**Theorem (Voiculescu)**

$S_{\mu \boxtimes \nu}(z) = S_\mu(z) \cdot S_\nu(z)$

# Free multiplicative normal distribution $\sigma_t$

## Definition (Bercovici, Voiculescu)

$S_{\sigma_t}(z) := \exp\left(t\left(z + \frac{1}{2}\right)\right)$

- $t \geq 0$: $\sigma_t \ldots$ measure on torus $\mathbb{T}$
  "free rolled-up normal distribution"
- $t \leq 0$: $\sigma_t \ldots$ measure on positive, real line $\mathbb{R}^+$
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## Properties of $\sigma_t$

- $\sigma_s \boxtimes \sigma_t = \sigma_{s+t} \quad s,t \geq 0$ or $s,t \leq 0$.
- $\sigma_t$ is $\boxtimes$-infinite divisible.

# Appearence of the measures $\sigma_t$

- Free, multiplicative central limit theorem
- Free, multiplicative Brownian Motion
- Limit distribution of some unitary random matrices

Philippe Biane. *Free Brownian motion, free stochastic calculus and random matrices*, 1997

# Moments of $\sigma_t$

**Theorem (Biane)**

*The moments of the measures $\sigma_t$ are for $n \geq 1$*

$$\mathrm{m_n}(\sigma_t) = \exp\left(-\frac{nt}{2}\right) \cdot \sum_{k=0}^{n-1} \frac{(-1)^k n^{k-1}}{k!} \binom{n}{k+1} t^k$$

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$$= \exp\left(-\frac{nt}{2}\right) \cdot {}_1\mathrm{F}_1(1-n;2;nt)$$

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# Confluent hypergeometric function

**Definition (Confluent hypergeometric series)**

$$ {}_1F_1(a;b;z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} z^n $$

with Pochhammer symbol

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**Remark:** $a \in -\mathbb{N}_0 \;\Rightarrow\; {}_1F_1(a;b;z)$ is a polynomial in $z$

$$
\begin{aligned}
\mathrm{m_n}(\sigma_t) &= \exp\left(-\frac{nt}{2}\right) \cdot \sum_{k=0}^{n-1} \frac{(-1)^k n^{k-1}}{k!} \binom{n}{k+1} t^k \\
&= \exp\left(-\frac{nt}{2}\right) \cdot {}_1\mathrm{F}_1(1-n; 2; nt) \\
&= \exp\left(\frac{nt}{2}\right) \cdot {}_1\mathrm{F}_1(1+n; 2; -nt)
\end{aligned}
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## Theorem

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$$\kappa_n = \frac{(-nt)^{n-1}}{n!} \exp\left(-\frac{nt}{2}\right).$$

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## Proof by induction on $n$

- moment-cumulant-formulas

$$\Rightarrow \quad \kappa_n = \left(\frac{(-n)^{n-1}}{n!} t^{n-1} + a_{n-2}^{(n)} t^{n-2} + \cdots + a_2^{(n)} t^2\right) e^{\frac{-nt}{2}}$$

- $\kappa_n(\sigma_{s+t}) = \kappa_n(\sigma_s \boxtimes \sigma_t) = \sum_{\pi \in NC(n)} \kappa_\pi(\sigma_s)\kappa_{K(\pi)}(\sigma_t)$

$$\Rightarrow \quad a_{n-2}^{(n)} = \cdots = a_2^{(n)} = 0$$

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$$ {}_1\mathrm{F}_1(1-n;2;nt) = \sum_{\pi \in NC(n)} \prod_{\substack{V \in \pi \\ k:=|V|}} \frac{(-kt)^{k-1}}{k!}$$

$$\frac{(-n)^{n-1}}{n!} t^{n-1} = \sum_{\pi \in NC(n)} \mu(\pi, 1_n) \prod_{\substack{V \in \pi \\ k:=|V|}} {}_1\mathrm{F}_1(1-k;2;kt)$$

# hypergeometric functions as (formal) moment sequences

$$m_n = {}_1F_1(1-n;2;nz) \quad \Leftrightarrow \quad \kappa_n = \frac{(-nz)^{n-1}}{n!}$$

$$m_n = {}_1F_1(1+n;2;nz) \quad \Leftrightarrow \quad \kappa_n = \frac{(nz)^{n-1}}{n!}\exp(nz)$$

## Theorem (Addition formulae for hypergeometric functions)

$$ {}_1F_1(1-n;2;x+y) = $$
$$ = \sum_{\pi \in NC(n)} \left( \prod_{\substack{V\in\pi \\ k:=|V|}} \frac{(-\frac{k}{n}x)^{k-1}}{k!} \cdot \prod_{\substack{V\in K(\pi) \\ l:=|V|}} {}_1F_1\left(1-l;2;\frac{l}{n}y\right) \right) $$

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**Proof:** $\mathrm{m_n}(\sigma_{s+t}) = \mathrm{m_n}(\sigma_s \boxtimes \sigma_t) = \sum_{\pi\in NC(n)} \kappa_\pi(\sigma_s)\cdot \mathrm{m}_{\mathrm{K}(\pi)}(\sigma_t)$

# free cumulants of the free log-normal distribution $(t < 0)$

**remark**

The Bercovici-Pata bijection fails, i.e. the log-normal distribution, which is *-infinite divisible, is not mapped to the free log-normal distribution.

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**open question**

Is the free log-normal distribution $\sigma_t$, $t < 0$, $\boxplus$-infinite divisible?

$\Leftrightarrow$ Is the sequence $m_n := \frac{(n+2)^n}{(n+1)!}$ positive definite, i.e. is $m_n$ a moment sequence?

# densitiy function of the free rolled-up normal distribution $\sigma_t$

$t > 0$ : $\sigma_t$ ...measure on the unit circle $\mathbb{T}$
$\tilde{\sigma}_t$ ... uncoiled measure on $[-\pi, \pi]$

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**Theorem**

*Let $t > 0$. The measure $\tilde{\sigma}_t$ is absolutely continuous. The density function is continuous, even, and monotone raising on $[-\pi, 0]$.*

$$\operatorname{supp}(\tilde{\sigma}_t) = \begin{cases} [-c, c] & 0 < t < 4 \\ [-\pi, \pi] & t \geq 4 \end{cases}$$

*where*

$$c = 2 \arctan\left(\sqrt{\frac{t}{4-t}}\right) + \sqrt{t\left(1 - \frac{t}{4}\right)}.$$

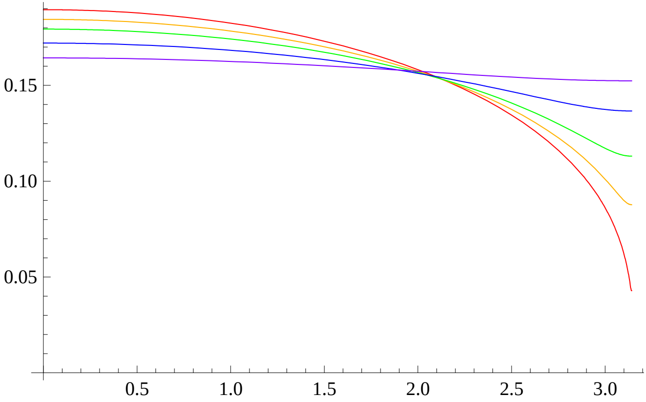
# approximation of the density function

Fourier series of the density function of $\tilde{\sigma}_t$

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \underbrace{\exp\left(-\frac{nt}{2}\right) {}_1F_1(1-n;2;nt)}_{=\mathrm{m_n}(\sigma_t)} \cos(nx)$$

- $0 < t \leq 4$: pointwise convergence
- $t > 4$: uniform convergence

# $\tilde{\sigma}_t$ densitiy function plots




$t = 4.1$ $t = 4.5$ $t = 5$ $t = 6$ $t = 8$

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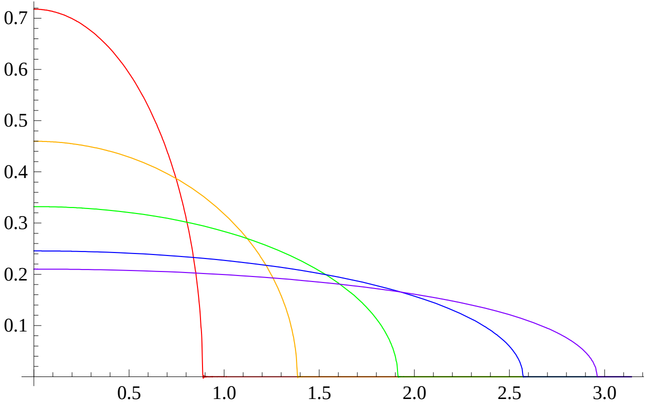


$t = 0.1$ $t = 0.5$ $t = 1$ $t = 2$ $t = 3$

# Notation

Let $\mu$ a measure on the torus $\mathbb{T}$.

uncoiled measure $\tilde{\mu}$ on $[-\pi, \pi]$

- $\tilde{\mu}(A) := \mu\left(\left\{e^{ix} \,|\, x \in A\right\}\right)$ for all $A \in \mathcal{B}((-\pi, \pi))$
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$0 < \varepsilon \leq 1$: Define measure $\mu^{(\varepsilon)}$ on $\mathbb{T}$ by dilatation of the uncoiled measure $\tilde{\mu}$

- $\widetilde{\mu^{(\varepsilon)}}([-\pi, -\varepsilon\pi) \cup (\varepsilon\pi, \pi]) = 0$
- $\widetilde{\mu^{(\varepsilon)}}(A) = \tilde{\mu}\left(\frac{1}{\varepsilon}A\right)$ for all $A \in \mathcal{B}([-\varepsilon\pi, \varepsilon\pi])$

## rolled-up normal distribution $\nu_t$

$$\int_{\mathbb{T}} f(z)\, d\nu_t := \int_{-\infty}^{\infty} f\left(e^{ix}\right) \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx$$

## Classical multiplicative CLT

Let $\mu$ a probability measure on $\mathbb{T}$ with $m_1(\tilde{mu}) = 0$ and let

$$\varepsilon_n = \frac{t}{\sqrt{n} \int_{-\pi}^{\pi} x^2\, d\tilde{\mu}(x)}.$$

Then the sequence

$$\mu_n = \underbrace{\mu^{(\varepsilon_n)} \circledast \cdots \circledast \mu^{(\varepsilon_n)}}_{n \text{ times}}$$

converge weakly to the rolled-up normal distribution $\nu_t$.

## Free multiplicative CLT

Let $\mu$ a probability measure on $\mathbb{T}$ with *all moments real* and let

$$\varepsilon_n = \frac{t}{\sqrt{n}\int_{-\pi}^{\pi} x^2\, d\tilde{\mu}(x)}.$$

Then the sequence

$$\mu_n = \underbrace{\mu^{(\varepsilon_n)} \boxtimes \cdots \boxtimes \mu^{(\varepsilon_n)}}_{n \text{ times}}$$

converge weakly to the free rolled-up normal distribution $\sigma_t$.

G. P. Chisyakov and F. Götze, *Limit theorems in free probability theory II*, Cent. Eur. J. Math., 2008