Colorings of bipartite graphs and polynomial functions on the set of Young diagrams

(joint work with Piotr Śniady)

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Uniwersytet Wrocławski

Bialgebras in Free Probability, Wien 2011

Representation theory of $\mathfrak{S}_n$

Definition

A partition λ is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. It can be represented by a Young diagram λ.

Fact

There is a one to one correspondence between

- *irreducible representations of $\mathfrak{S}_n$;*
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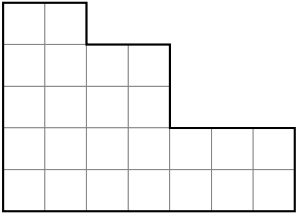

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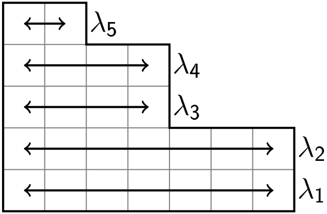

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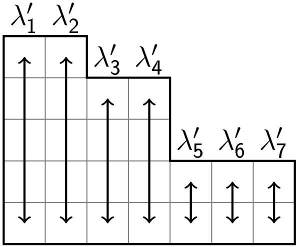

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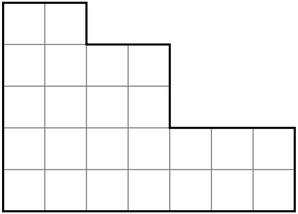

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Normalized characters

Character as a function on the set of Young diagrams $\mathbb{Y}$:

- Fix our favorite permutation $\pi \in \mathfrak{S}_k$.
- Let λ has n boxes. For $n \geq k$ we have a natural embedding $\mathfrak{S}_k \hookrightarrow \mathfrak{S}_n$ hence we can consider π as an element of $\mathfrak{S}_n$.
- Character is a function on $\mathbb{Y}$ defined by:

$$\chi_\pi(\lambda) = \begin{cases} \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \dfrac{\operatorname{Tr}(\rho^\lambda(\pi))}{\text{dimension of } \rho^\lambda} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

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Simple operations on Young diagrams

- Dilation $D_s\lambda = (\underbrace{s\lambda_1, \dots, s\lambda_1}_{s \text{ factors}}, \dots, \underbrace{s\lambda_k, \dots, s\lambda_k}_{s \text{ factors}})$ of λ by s.

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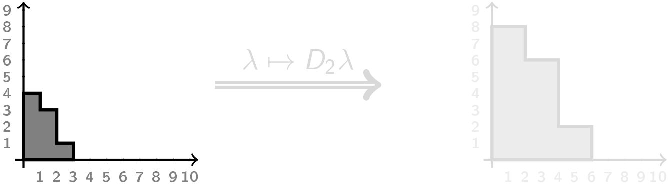

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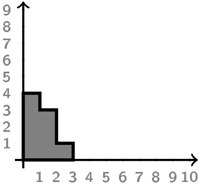

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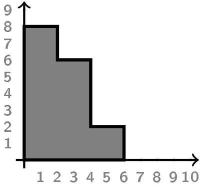

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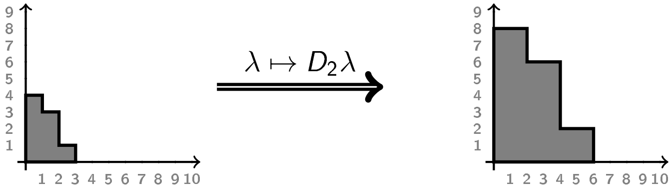

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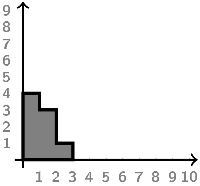

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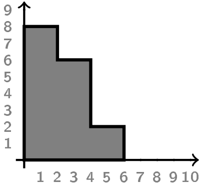

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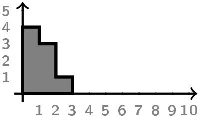

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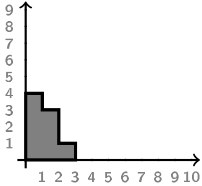

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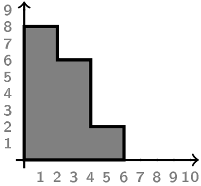

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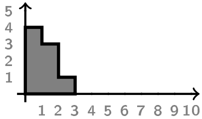

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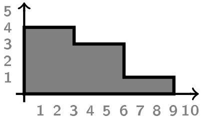

Other functions on the set of Young diagrams

- Free cumulants $R_k(\lambda)$ are relatively simple functions given by
$$R_k(\lambda) = \lim_{s\to\infty} \frac{1}{s^k} \Sigma_{(12\ldots k-1)}(D_s\lambda);$$
Advantages: good approximation of normalized characters: $R_k(\lambda) \approx \Sigma_{(12\ldots k-1)}(\lambda)$;
- Fundamental functionals of shape $S_k(\lambda)$ given by
$$S_k(\lambda) = (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2}\, dx\, dy$$
gives an information about the shape of λ. Advantages: very useful and powerful in differential calculus on $\mathbb{Y}$
- Jack characters are functions with additional parameter α (related to α-anisotropic Young diagrams). They generalize normalized characters and they are quite mysterious.

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Relations between them

- They generate the same algebra $\mathcal{P}$ of polynomial functions on $\mathbb{Y}$ (studied by Kerov and Olshanski);
- $\mathcal{P}$ is isomorphic to subalgebra of partial permutations - related to computing connection coefficients or studying multiplication of conjugacy classes in the symmetric groups;
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Number of colorings

Definition

- Let G be a bipartite graph.
- Let $V = V_\circ \sqcup V_\bullet$ be a set of vertices.
- Any function $h : V \to \mathbb{N}$ is called a coloring of a graph G.
- A coloring h is compatible with Young diagram λ if $(h(v_1), h(v_2)) \in \lambda$ whenever $(v_1, v_2) \in V_\circ \times V_\bullet$ is an edge in G.
- We will define a function $N_G(\lambda)$ as a number of colorings of G which are compatible with λ.

Let us show some examples:

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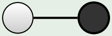
- Let $G =$  . Then $N_G(\lambda) = \sum \lambda_i = \sum \lambda'_i = |\lambda|$.
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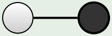
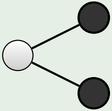
- Let $G =$  . Then $N_G(\lambda) = \sum \lambda_i = \sum \lambda'_i = |\lambda|$.
- Let $G =$  . Then $N_G(\lambda) = \sum \lambda'^2_i$.
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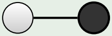
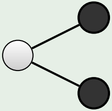
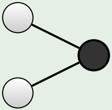
- Let $G =$  . Then $N_G(\lambda) = \sum \lambda_i = \sum \lambda'_i = |\lambda|$.
- Let $G =$  . Then $N_G(\lambda) = \sum {\lambda'_i}^2$.
- Let $G =$  . Then $N_G(\lambda) = \sum \lambda_i^2$.

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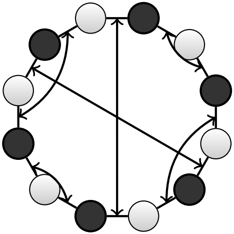

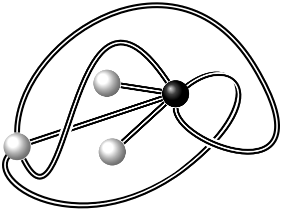

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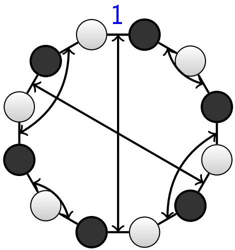

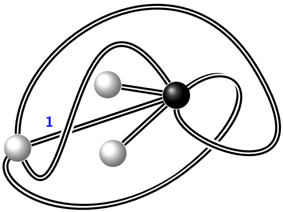

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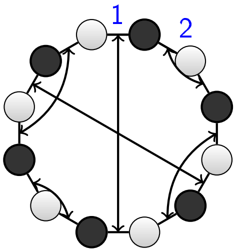

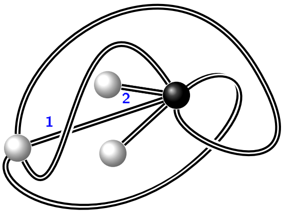

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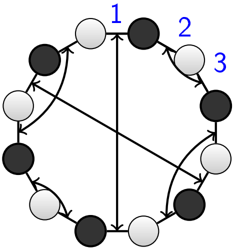

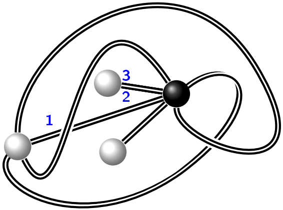

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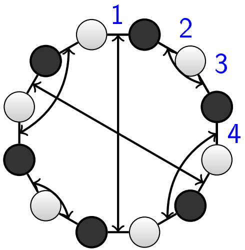

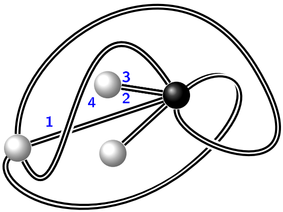

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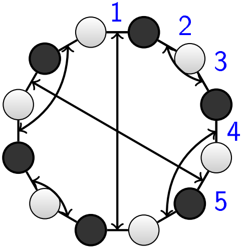

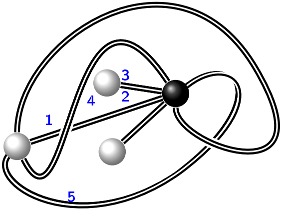

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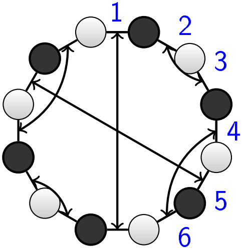

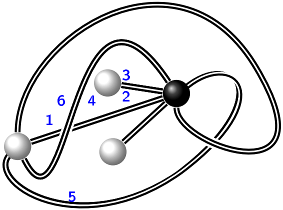

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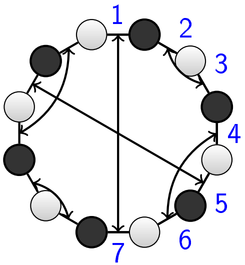

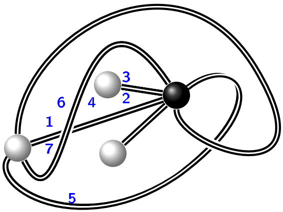

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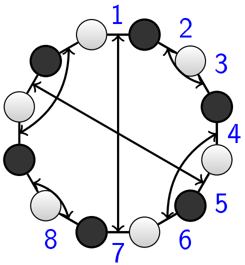

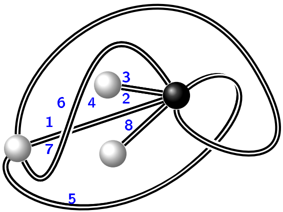

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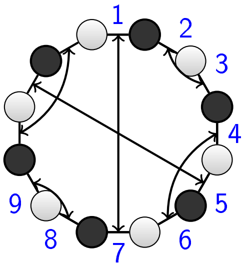

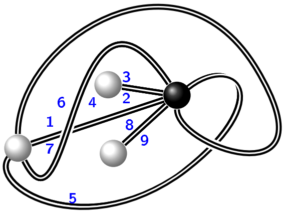

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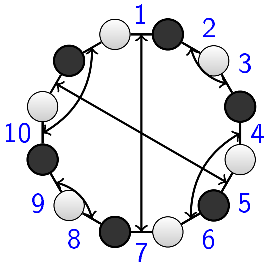

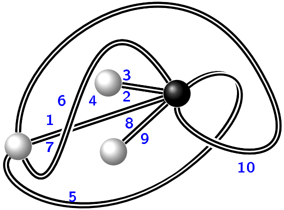

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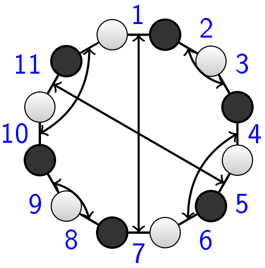

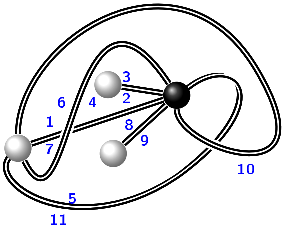

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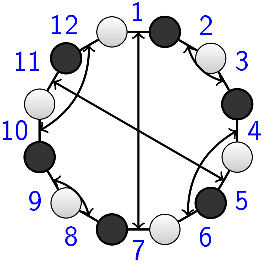

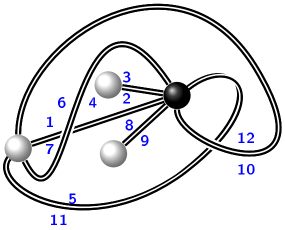

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$$\Sigma_\mu = \sum_{\mathcal{M}} (-1)^{|\mu| - |V_\circ(\mathcal{M})|} \; N_{\mathcal{M}},$$

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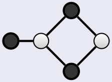
Let G be a bipartite graph: 

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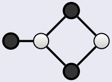
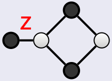
Let G be a bipartite graph: 

- $\partial_z G =$ 

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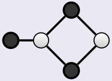
Let G be a bipartite graph: 

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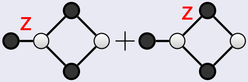

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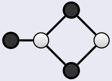
Let G be a bipartite graph: 

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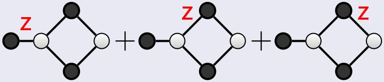

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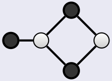
Let G be a bipartite graph: 

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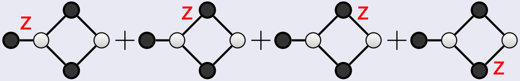

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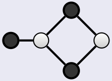
Let G be a bipartite graph: 

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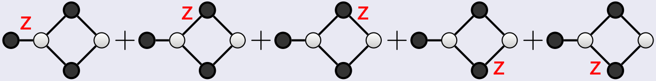

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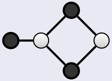
Let G be a bipartite graph: 

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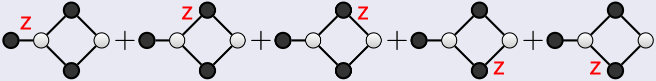
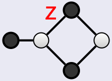


Let $(G, z) =$ 

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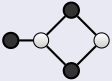
Let G be a bipartite graph: 

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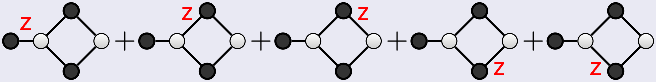
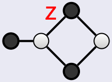
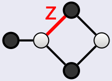


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

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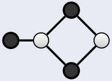
Let G be a bipartite graph: 

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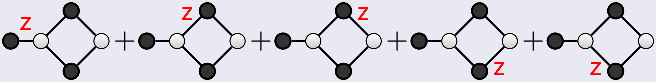
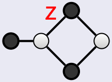
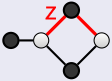


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

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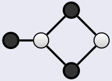
Let G be a bipartite graph: 

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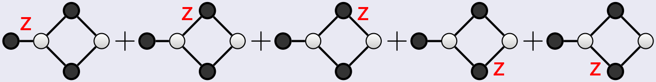
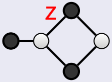
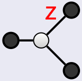


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

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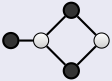
Let G be a bipartite graph: 

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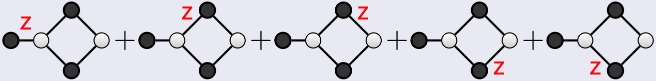
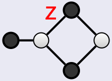
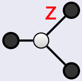
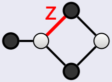


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

- $\partial_y (G, z) =$ 

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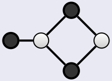
Let G be a bipartite graph: 

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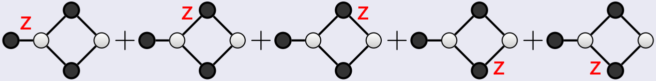
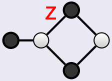
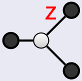
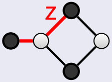


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

- $\partial_y (G, z) =$ 

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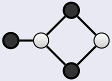
Let G be a bipartite graph: 

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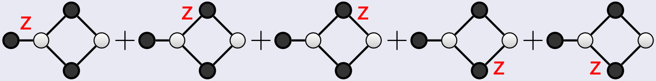
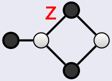
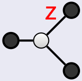
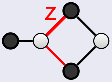


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

- $\partial_y (G, z) =$ 

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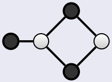
Let G be a bipartite graph: 

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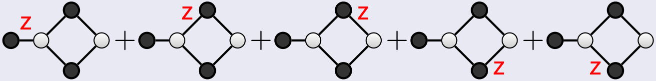
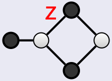
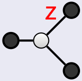
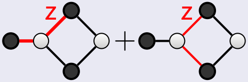


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

- $\partial_y (G, z) =$ 

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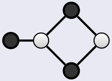
Let G be a bipartite graph: 

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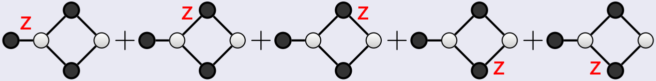
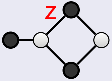
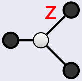
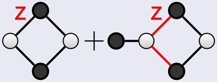


Let $(G, z) =$ 

- $\partial_x(G, z) =$ 

- $\partial_y(G, z) =$ 

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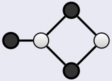
Let G be a bipartite graph: 

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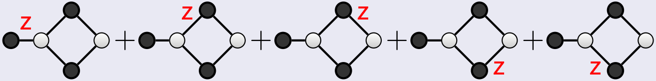
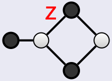
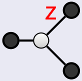
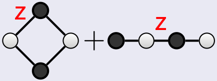


Let $(G, z) =$ 

- $\partial_x (G, z) =$ 

- $\partial_y (G, z) =$ 

Main theorem

Theorem (D., Śniady)

Let $\mathcal{G}$ be a linear combination of bipartite graphs such that

$$(\partial_x + \partial_y)\,\partial_z \mathcal{G} = 0.$$

Then $\lambda \mapsto N_{\mathcal{G}}(\lambda)$ is a polynomial function on the set of Young diagrams.

Corollary

$$\sum_{\mathcal{M}} (-1)^{|V_\circ(\mathcal{M})|}\; N_{\mathcal{M}}, \tag{1}$$

where the summation is over all labeled bipartite maps with the face type μ is a polynomial function on $\mathbb{Y}$.

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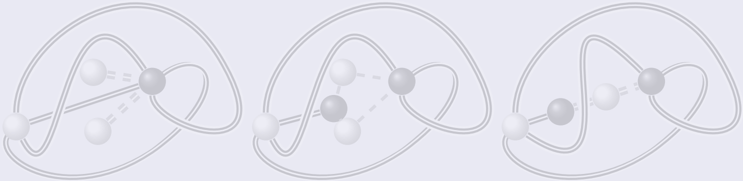
where the summation is over all (not only oriented) labeled bipartite maps with the face type μ is a polynomial function on $\mathbb{Y}$.

Proof of the corollary

Proof.

By the Main Theorem it suffices to show that $(\partial_x + \partial_y)\partial_z \left(\sum_{\mathcal{M}} (-1)^{|V_\circ(\mathcal{M})|} \mathcal{M}\right) = 0$. Let us look at ∂_x:



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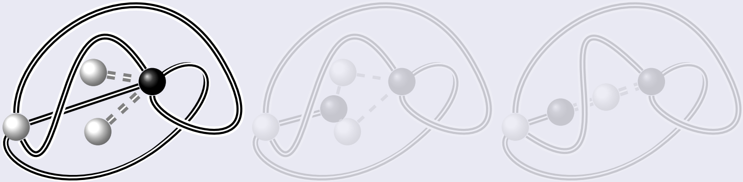

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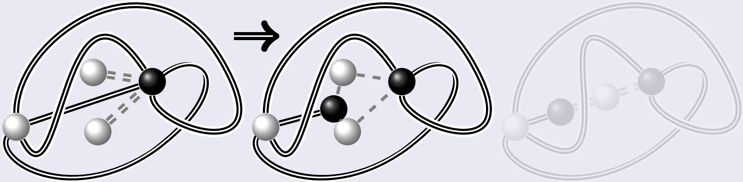

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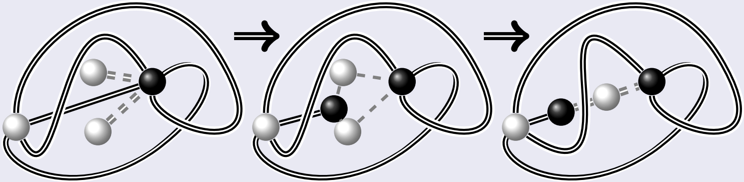

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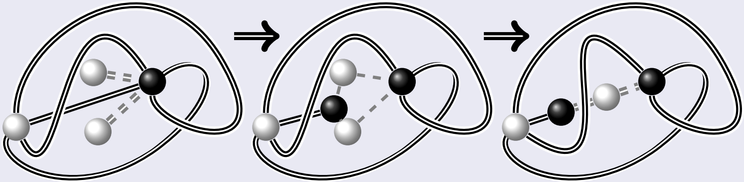

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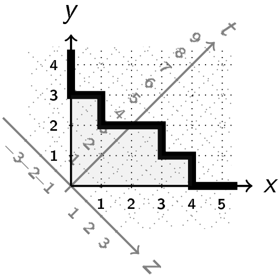

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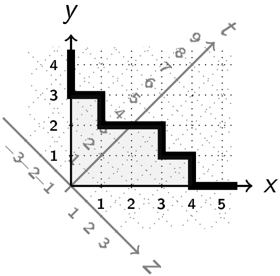

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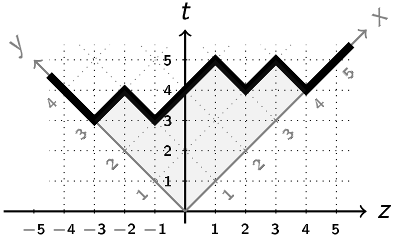

Young diagrams as functions

We want to make a differential calculus on $\mathbb{Y}$.

Problem

Young diagrams are very discrete.

Solution

We can define generalized Young diagrams as continous objects!

Definition

A generalized Young diagram is a function $\omega : \mathbb{R}_+ \to \mathbb{R}$ such that:

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Let F be a function on $\mathbb{Y}$.

Problem

How quickly the value of $F(\lambda)$ would change if we change the shape of λ by adding infinitesimal boxes with content equal to $z = z_0$?

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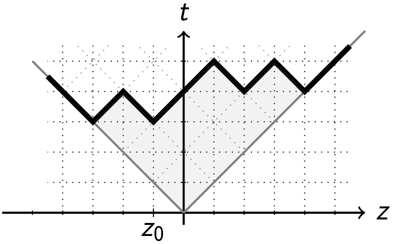

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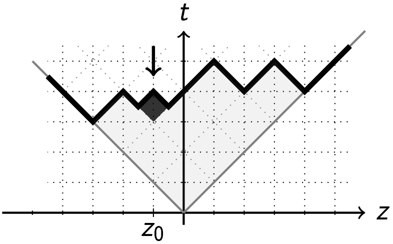

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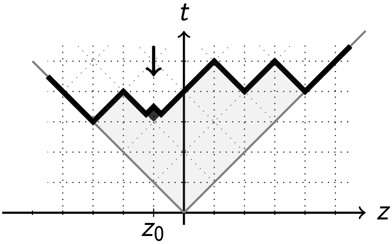

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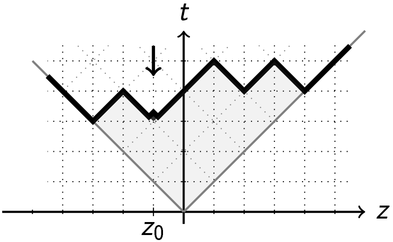

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Solution

Content-derivative $\partial_{C_z} F(\lambda)$ will measure it!

Details of the proof

Proof.

Let G be a bipartite graph and $\mathcal{G}$ be a linear combination of bipartite graphs. Then:

- $\partial_{C_z} N_G(\lambda) = N_{\partial_z G}(\lambda)$,
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- $\frac{d}{dz}\partial_{C_z} N_G(\lambda) = \frac{\omega'(z)+1}{2} N_{\partial_x \partial_z G}(\lambda) + \frac{\omega'(z)-1}{2} N_{\partial_y \partial_z G}(\lambda)$,
- $\frac{d}{dz}\partial_{C_z} N_{\mathcal{G}}(\lambda) = \frac{1}{2} N_{(\partial_x - \partial_y)\partial_z G}$,
- $\frac{d^i}{dz^i}\partial_{C_z} N_{\mathcal{G}}(\lambda) = \frac{1}{2^i} N_{(\partial_x - \partial_y)^i \partial_z G}$,
- $z \mapsto \partial_{C_z} N_{\mathcal{G}}(\lambda)$ is polynomial,
- $[z^k]\partial_{C_z} N_{\mathcal{G}}(\lambda)$ is a polynomial function on $\mathbb{Y}$,
- $N_{\mathcal{G}}(\lambda)$ is a polynomial function on $\mathbb{Y}$.

$\square$

Characterization of Jack shifted symmetric functions

Jack shifted symmetric functions $J_\mu^{(\alpha)}$ with parameter α came from symmetric functions theory. They are characterized by three conditions:

- $J_\mu^{(\alpha)}(\mu) \neq 0$ and for each Young diagram $\lambda \neq \mu$ such that $|\lambda| \leq |\mu|$ we have $J_\mu^{(\alpha)}(\lambda) = 0$;
- $J_\mu^{(\alpha)}$ has degree equal to $|\mu|$ (regarded as a shifted symmetric function);
- The function $\lambda \mapsto J_\mu^{(\alpha)}\left(\frac{1}{\alpha}\lambda\right)$ is a polynomial function.

Jack characters $\Sigma_\pi^{(\alpha)}$ are given by:

$$J_\mu^{(\alpha)}(\lambda) = \sum_{\pi \vdash |\mu|} n_\pi^{(\alpha)} \, \Sigma_\pi^{(\alpha)}(\mu) \, \Sigma_\pi^{(\alpha)}(\lambda).$$

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- Jack characters generalize normalized characters:

$$\Sigma_\pi(\lambda) = \Sigma_\pi^{(1)}(\lambda)$$

- How to express Jack characters in terms of N_G?
- For some α first two conditions are easy to verify.

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$$\Sigma_\mu^{(2)} = (-1)^{|\mu|} \sum_{\mathcal{M}} (-2)^{|V_\circ(\mathcal{M})|} \; N_{\mathcal{M}},$$

where the summation is over all labeled bipartite (not only oriented) maps with the face type μ.

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- $N_G(\lambda)$ is a number of colorings of vertices of G (by natural numbers) which are compatible with λ.

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- $N_G(\lambda)$ is a number of colorings of edges of G (by boxes of λ) such that whenever $e_1 \cap e_2 \in V_\circ$ ($V_\bullet$ resp.), then $h(e_1)$ and $h(e_2)$ are in the same column (row resp.) of λ.

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- Let $\tilde{N}_G(\lambda)$ is a number of injective colorings of edges of G as above.
- We have that:

$$\sum_{\mathcal{M}} (-1)^{|V_\circ(\mathcal{M})|} N_{\mathcal{M}} = \sum_{\mathcal{M}} (-1)^{|V_\circ(\mathcal{M})|} \tilde{N}_{\mathcal{M}}$$

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Problem

For which polynomials $f_{\mathcal{M}} \in \mathbb{Q}[x]$ we have

$$\sum_{\mathcal{M}} (-\alpha)^{|V_\circ(\mathcal{M})|} f_{\mathcal{M}}(\alpha) N_{\mathcal{M}} = \sum_{\mathcal{M}} (-\alpha)^{|V_\circ(\mathcal{M})|} f_{\mathcal{M}}(\alpha) \tilde{N}_{\mathcal{M}}$$

for all $\alpha \in \mathbb{R}_+$?