Second-order Freeness in the Real Case

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Outline

Definition

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Let $A_1, \ldots, A_n \subseteq A$. We say that $A_1, \ldots, A_n$ are *free* if, for any centred $a_1, \ldots, a_m$ with $a_i \in A_{k_1}$ and $k_1 \neq k_2 \neq \cdots \neq k_m$, we have $\varphi(a_1 \cdots a_m) = 0$.

Definition

Let $X_{c,N} : \Omega \rightarrow M_{N\times N}(\mathbb{C})$ be random $N \times N$ matrices for $1 \leq c \leq n$. The $X_{c,N}$ are *asymptotically free* if, for alternating word in the colours w and $A_{k,N}$ in the algebra generated by $X^{(\pm\varepsilon)}_{w(k),N}$,

$$\lim_{N\rightarrow\infty} \mathbb{E}\left(\mathrm{tr}\left(\left(A_{1,N} - \mathbb{E}\left(\mathrm{tr}\left(A_{1,N}\right)\right)\right)\cdots\right.\right.$$
$$\left.\left.\left(A_{r,N} - \mathbb{E}\left(\mathrm{tr}\left(A_{m,N}\right)\right)\right)\right)\right) = 0.$$

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Independent copies of many important matrix ensembles are asymptotically free.

Definition

A *second-order probability space* is a noncommutative probability space (A, φ) equipped with a bilinear function $\rho : A \times A \to \mathbb{C}$ which is tracial in each argument and has $\rho(1_A, a) = \rho(a, 1_A) = 0$ for all $a \in A$.

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The function ρ models the rescaled covariance of traces.

Definition

Subspaces $A_1, \ldots, A_n \subseteq A$ are said to be *second-order (complex) free* if they are free and (taking indices modulo the appropriate range),

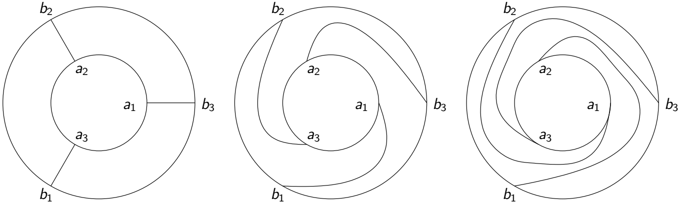
$$\rho\left(a_1 \cdots a_p, b_1 \cdots b_p\right) = \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi\left(a_i b_{k-i}\right)$$

when the $a_1, \ldots, a_p$ and the $b_1, \ldots, b_p$ are centred and cyclically alternating, and

$$\rho\left(a_1 \cdots a_p, b_1 \cdots b_q\right) = 0$$

when $p \neq q$ and the $a_1, \ldots, a_p$ and the $b_1, \ldots, b_q$ are centred and either cyclically alternating or consist of a single term.

The pairing of terms can be represented diagrammatically in spoke diagrams:



Definition

Random matrices $X_{c,N} : \Omega \to (\mathbb{C})$ are *asymptotically real second-order free* if, for cyclically alternating (or length 1) words in the colours v and w and for $A_{k,N}$ in the algebra generated by $X^{(\pm\varepsilon)}_{v(k),N}$ and $B_{k,N}$ in the algebra generated by $X^{(\pm\varepsilon)}_{w(k),N}$, the expression

$$\lim_{N\to\infty} k_2 \left(\mathrm{Tr}\left(\left(A_1 - \mathbb{E}\left(\mathrm{tr}\left(A_1\right)\right)\right)\cdots\left(A_p - \mathbb{E}\left(\mathrm{tr}\left(A_p\right)\right)\right)\right),\right.$$
$$\left.\mathrm{Tr}\left(\left(B_1 - \mathbb{E}\left(\mathrm{tr}\left(B_1\right)\right)\right)\cdots\left(B_q - \mathbb{E}\left(\mathrm{tr}\left(B_q\right)\right)\right)\right)\right)$$

is equal to 0 whenever $p \neq q$,

Definition (cont'd)

and equal to

$$\sum_{k=0}^{p-1} \prod_{i=1}^{p} \left(\mathbb{E}\left(\mathrm{tr}\left(A_1 B_{k-i}\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right) \mathbb{E}\left(\mathrm{tr}\left(B_{k-i}\right)\right) \right)$$

whenever $p = q \geq 2$.

Let:

- $\mathrm{tr} := \frac{1}{N}\mathrm{Tr}$,
- $n_1, \ldots, n_r$ positive integers, $n := n_1 + \cdots + n_r$,
- $A^{(1)} = A$, $A^{(-1)} = A^T$,
- $[n] = \{1, \ldots, n\}$,
- $\varepsilon : [n] \to \{1, -1\}$,
- $\delta_\varepsilon : k \mapsto (-1)^{\varepsilon(k)} k$.

Let $X : \Omega \to M_{M \times N}(\mathbb{R})$ be a random matrix with $X_{ij} = \frac{1}{\sqrt{N}} f_{ij}$.

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We wish to calculate expressions of the form

$$\mathbb{E}\left(\mathrm{tr}\left(X^{(\varepsilon(1))} D_1 \cdots X^{(\varepsilon(n_1))} D_{n_1}\right) \cdots \mathrm{tr}\left(X^{(\varepsilon(n_1+\cdots+n_{r-1}+1))} D_{n_1+\cdots+n_{r-1}+1} \cdots X^{(\varepsilon(n))} D_n\right)\right).$$

For $\gamma = (c_1, \ldots, c_{n_1}) \cdots (c_{n_1+\cdots+n_{r-1}}, \ldots, c_n) \in S_n$, we define:

$$\mathrm{Tr}_\gamma (A_1, \ldots, A_n) := \mathrm{Tr}\left(A_{c_1} \cdots A_{c_{n_1}}\right) \cdots \mathrm{Tr}\left(A_{c_{n_1+\cdots+n_{r-1}}} \cdots A_{c_n}\right).$$

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Then

$$\mathrm{Tr}_\gamma (A_1, \ldots, A_n) = \sum_{1 \leq i_1, \ldots, i_n \leq N} A_{i_1 i_{\gamma(1)}} \cdots A_{i_n i_{\gamma(n)}}.$$

Then our expression is:

$$\sum_{\substack{1\leq\iota_1^+,\ldots,\iota_n^+\leq M\\1\leq\iota_1^-,\ldots,\iota_n^-\leq N}} N^{-\#\gamma-n} D^{(1)}_{\iota_1^{-\varepsilon(1)}\iota_{\gamma(1)}^{\varepsilon(\gamma(1))}}\cdots D^{(n)}_{\iota_n^{-\varepsilon(n)}\iota_{\gamma(n)}^{\varepsilon(\gamma(n))}}\mathbb{E}\left(f_{\iota_1^+\iota_1^-}\cdots f_{\iota_n^+\iota_n^-}\right).$$

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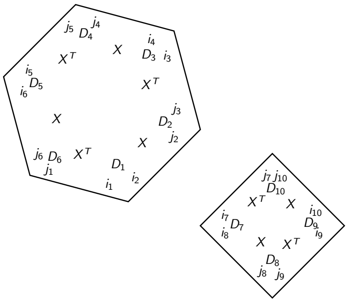
We construct a face for each trace, with the D_k matrices as vertices and the random matrices as edges, arranged cyclically as they appear in the traces.

To calculate the expression

$$\mathbb{E}\left(\mathrm{tr}\left(X^T D_1 X D_2 X^T D_3 X D_4 X^T D_5 X D_6\right)\right.$$
$$\left.\mathrm{tr}\left(X^T D_7 X D_8 X^T D_9 X D_{1)}\right)\right)$$

we construct:



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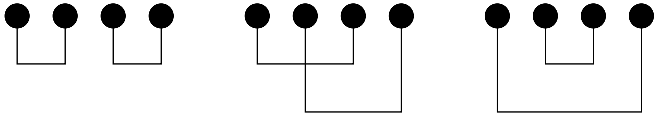
Theorem

Let $\{f_\lambda : \lambda \in \Lambda\}$, for some index set Λ, be a centred Gaussian family of random variables. Then for $i_1, \ldots, i_n \in \Lambda$,

$$\mathbb{E}\left(f_{i_1} \cdots f_{i_n}\right) = \sum_{\mathcal{P}_2(n)} \prod_{\{k,l\} \in \mathcal{P}_2(n)} \mathbb{E}\left(f_{i_k} f_{i_l}\right).$$

There are three pairings on 4 elements:



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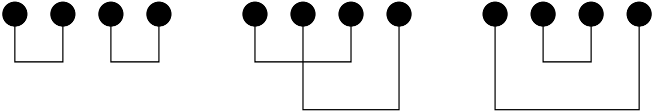

If X_1, X_2, X_3, X_4 are components of a multivariate Gaussian random variable, then

$$\mathbb{E}(X_1X_2X_3X_4) = \mathbb{E}(X_1X_2)\,\mathbb{E}(X_3X_4) + \mathbb{E}(X_1X_3)\,\mathbb{E}(X_2X_4) + \mathbb{E}(X_1X_4)\,\mathbb{E}(X_2X_3).$$

Then, for a pairing $\pi \in \mathcal{P}_2(n)$:

$$\prod_{\{k,l\}} \mathbb{E}\left(f_{i_k j_k} f_{i_l j_l}\right) = \begin{cases} 1, & \text{if } i_k = i_l \text{ and } j_k = j_l \text{ for all } \{k,l\} \in \pi \\ 0, & \text{otherwise} \end{cases} .$$

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Our expression becomes:

$$\sum_{\substack{1 \leq \iota_1^+, \ldots \iota_n^+ \leq M \\ 1 \leq \iota_1^-, \ldots, \iota_n^- \leq N}} \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \iota_k^\pm = \iota_l^\pm : \{k,l\} \in \pi}} N^{-\#\gamma - n} D^{(1)}_{\iota_1^{-\varepsilon(1)} \iota_{\gamma(1)}^{\varepsilon(\gamma(1))}} \cdots D^{(n)}_{\iota_n^{-\varepsilon(n)} \iota_{\gamma(n)}^{\varepsilon(\gamma(n))}} .$$

Reversing the order of summation, these become constraints on the values of the indices:

$$\sum_{\pi\in\mathcal{P}_2(n)} \sum_{\substack{1\leq\iota_1^+,\ldots\iota_n^+\leq M\\ 1\leq\iota_1^-,\ldots,\iota_n^-\leq N\\ \iota_k^\pm=\iota_l^\pm:\{k,l\}\in\pi}} N^{-\#\gamma-n} D^{(1)}_{\iota_1^{-\varepsilon(1)}\iota_{\gamma(1)}^{\varepsilon(\gamma(1))}}\cdots D^{(n)}_{\iota_n^{-\varepsilon(n)}\iota_{\gamma(n)}^{\varepsilon(\gamma(n))}}.$$

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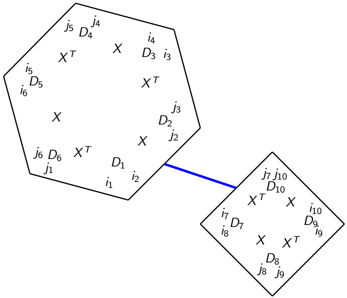
We represent these constraints by gluing the edges so that constrained vertices are adjacent.

We note that if one term comes from X and the other from X^T, the identification is untwisted.



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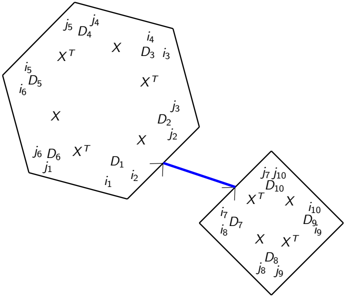
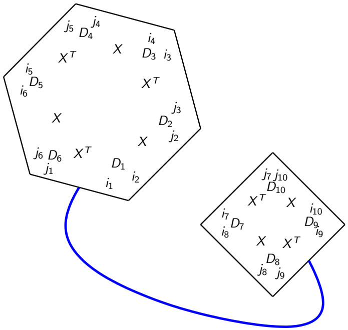


If both terms come from X or both from X^T, the identification is twisted.



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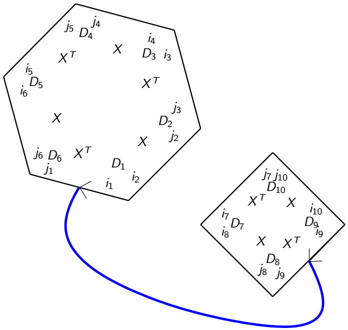
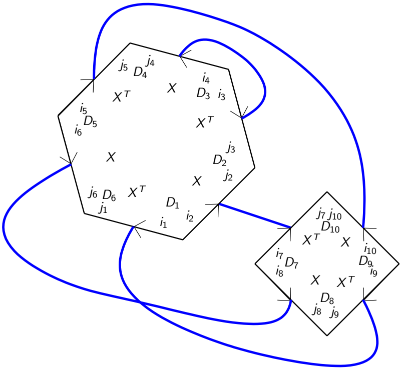
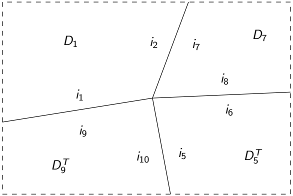


We construct a (possibly nonorientable) surface this way:



We note that the constraints appearing on the D_k matrices around a vertex are exactly those that would appear in a trace.



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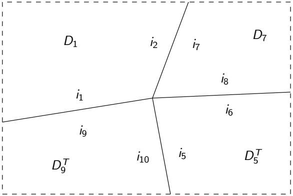

Corners which appear upside down become transposed matrices.

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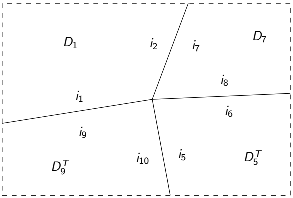

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This vertex contributes a factor of $\mathrm{Tr}\left(D_1 D_7 D_5^T D_9^T\right)$.

Each vertex gives us a trace, and hence a factor of N when normalized.

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Since the number of faces and edges are fixed, highest order terms are those with the highest Euler characteristic (typically spheres or collections of spheres).

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If there is no relative orientation of the faces such that there are no twisted identfications, the surface is nonorientable, so highest order terms must have a relative orientation of the faces in which none of the edge-identifications are twisted.

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This construction also works if π encodes hyperedges.

To extend this construction to unoriented surfaces, we construct the orientable two-sheeted covering space (the surface experience by someone *on* the surface rather than *within* it).

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An untwisted edge-identification connects front to front and back to back, while a twisted edge-identification connects front to back and back to front.

We label the front sides with positive integers and the corresponding back sides with negative integers.

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Vertex information is given by $\gamma_-^{-1}\pi^{-1}\gamma_+$.

In the example,

$$\pi = (1,-9)\,(-1,9)\,(2,7)\,(-2,-7)\,(3,4)\,(-3,-4)\,(5,10)\,(-5,-10)\,(6,-8)\,(-6,8)\,.$$

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This diagram contributes the term:

$$N^{-12}\mathrm{Tr}\left(D_1 D_7 D_5^T D_9^T\right)\mathrm{Tr}\left(D_2 D_4 D_{10}\right)\mathrm{Tr}\left(D_3\right)\mathrm{Tr}\left(D_6 D_8^T\right)$$

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Thus

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(Z^{(\varepsilon(1))}, \ldots, Z^{(\varepsilon(n))}\right)\right) = \sum_{\pi \in \{\rho\delta\rho : \rho \in \mathcal{P}_2(n)\}} N^{\chi(\gamma, \delta_{\varepsilon}\pi\delta_{\varepsilon}) - \#(\gamma)}.$$

If we expand out the GOE matrix $T := X + X^T$, we get

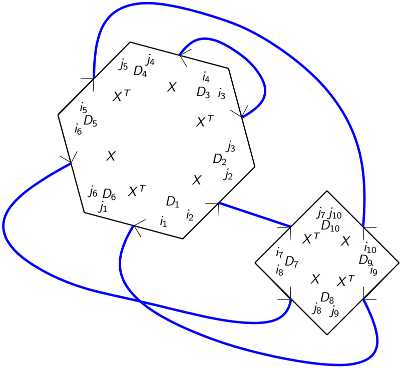
$$\mathbb{E}\left(\mathrm{tr}\left(T \cdots T\right) \cdots \mathrm{tr}\left(T \cdots T\right)\right) = \sum_{\varepsilon:\{1,\ldots,n\}\rightarrow\{1,-1\}} \frac{1}{\sqrt{2}} \mathbb{E}\left(\mathrm{tr}\left(X^{(\varepsilon(1))} \cdots X^{(\varepsilon(n_1))}\right) \cdots \mathrm{tr}\left(X^{(\varepsilon(n_1+\cdots+n_{r-1}+1))} \cdots X^{(\varepsilon(n))}\right)\right).$$

If we collect terms, this is equivalent to summing over all edge-identifications.

Thus

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(T,\ldots,T\right)\right)=\sum_{\pi\in PM(\pm[n])\cap\mathcal{P}_2(\pm[n])}N^{\chi(\gamma,\pi)-\#(\gamma)}.$$

With Wishart matrices $W := X^T D_k X$, we can collapse the edges corresponding to each matrix to a single edge. We can think of the connecting blocks as (possibly twisted) hyperedges.



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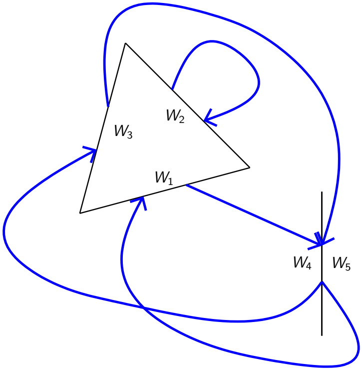

Thus:

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(W_1, \cdots, W_n\right)\right) = \sum_{\pi \in PM([n])} N^{\chi(\gamma,\pi)-\#(\gamma)} \mathrm{tr}_{\pi}\left(D_1, \ldots, D_n\right).$$

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We wish to calculate expressions of the form:

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(O^{\varepsilon(1)}D_1,\ldots,O^{\varepsilon(n)}D_n\right)\right) = \sum_{1\leq\iota_1^{\pm},\ldots,\iota_n^{+}\leq N} N^{-\#\gamma} D^{(1)}_{\iota_1^{-\varepsilon(1)}\iota_{\gamma(1)}^{\varepsilon(\gamma(1))}}\cdots D^{(n)}_{\iota_n^{-\varepsilon(n)}\iota_{\gamma(n)}^{\varepsilon(\gamma(n))}} \mathbb{E}\left(O_{\iota_1^{+}\iota_1^{-}}\cdots O_{\iota_n^{+}\iota_n^{-}}\right).$$

Theorem (Collins and Śniady, 2006)

If O is a Haar-distributed orthogonal matrix, then

$$\mathbb{E}\left(O_{i_1 j_1} \cdots O_{i_n j_n}\right) = \sum_{\substack{(\pi_1,\pi_2)\in\mathcal{P}_2(n)^2 \\ i_k=i_l:\{k,l\}\in\pi_1 \\ j_k=j_l:\{k,l\}\in\pi_2}} \langle \pi_1, \mathrm{Wg}\pi_2 \rangle$$

where $\mathrm{Wg} : \mathbb{C}\left[\mathcal{P}_2\left(n\right)\right] \rightarrow \mathbb{C}\left[\mathcal{P}_2\left(n\right)\right]$ is the Weingarten function for orthogonal matrices.

Definition

Define $\tilde{\Phi} : \mathbb{C}\left[\mathcal{P}_2\left(n\right)\right] \to \mathbb{C}\left[\mathcal{P}_2\left(n\right)\right]$ by letting the entry corresponding to the pairings π_1 and π_2 be

$$\langle \pi_1, \tilde{\Phi}\pi_2 \rangle = N^{\#\text{ of loops in }\pi_1 \cup \pi_2}.$$

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$$\left\langle \pi_1, \tilde{\Phi}\pi_2 \right\rangle = N^{\#\text{ of loops in }\pi_1 \cup \pi_2}.$$

Define the Weingarten function

$$\mathrm{Wg} := \tilde{\Phi}^{-1}.$$

Let $\pi_1 = \{\{1,9\},\{2,7\},\{3,4\},\{5,10\},\{6,8\}\}$ and $\pi_2 = \{\{1,10\},\{2,6\},\{3,4\},\{5,9\},\{7,8\}\}$.

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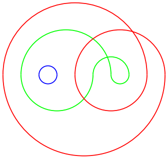
Then $\pi_1 \cup \pi_2$ has 3 loops:



so $\langle \pi_1, \tilde{\Phi}\pi_2 \rangle = N^3$.

We find that the constraints on the D_k matrices are given by the cycles of the permutation $\gamma_-^{-1}\delta_\varepsilon\pi_1\delta\pi_2\delta_\varepsilon\gamma_+$

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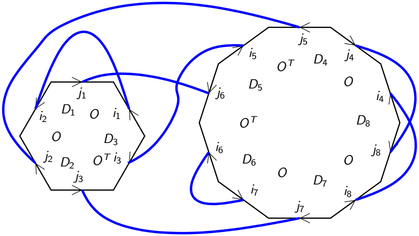
This suggests a diagram in which the two indices of O have separate edges, identified according to the appropriate π_i (or hyperedges of the form $\delta_\varepsilon\pi_2\delta\pi_1\delta_\varepsilon$).

If we are calculating

$$\mathbb{E}\left(\mathrm{tr}\left(OD_1OD_2O^TD_3\right)\cdots\mathrm{tr}\left(OD_4O^TD_5O^TD_6OD_7OD_8\right)\right)$$

and $\pi_1 = (1,2)(3,5)(4,8)(6,7)$ and $\pi_2 = (1,6)(2,5)(3,7)(4,8)$, we construct:



We note that there are extraneous vertices containing the O matrices.

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Theorem (Collins and Śniady, 2006)

If $\pi_1 \cup \pi_2$ has blocks of size $n_1, \ldots, n_k$, the leading term of the Weingarten function $\langle \pi_1, \mathrm{Wg}\pi_2 \rangle$ is

$$\prod_{i=1}^{k} (-1)^{n_i/2-1} \, c_{n_i/2-1} N^{\#\pi_1\pi_2/2-n}$$

where $c_m := \frac{1}{m+1}\binom{2m}{m}$ is the mth Catalan number.

At leading order, each of these extraneous vertices contributes a factor of $(-1)^{k/2-1} c_{k/2-1} N$, where k is the degree of the vertex.

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Thus:

$$\mathbb{E}\left(\mathrm{tr}_\gamma\left(O^{\varepsilon(1)},\ldots,O^{\varepsilon(n)}\right)\right)$$
$$=\sum_{(\pi_1,\pi_2)\in\mathcal{P}_2(n)^2} N^{\chi(\gamma,\delta_\varepsilon\pi_2\delta\pi_1\delta_\varepsilon)-\#(\gamma)}\left(N^{\#(\pi_1\pi_2)-n}\langle\pi_1\mathrm{Wg}\pi_2\rangle\right)$$

Definition

The *nth mixed moment* of (classical) random variables $X_1, \ldots, X_n$ is an n-linear function defined to be the expectation of their product:

$$a_n \left(X_1, \ldots, X_n \right) := \mathbb{E} \left(X_1 \cdots X_n \right).$$

Definition

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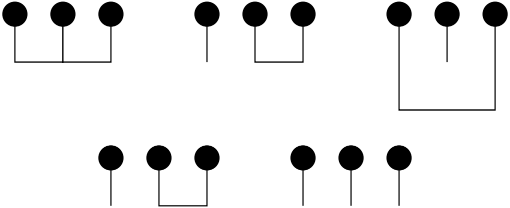
We define the k_i to satisfy the moment-cumulant formula:

$$a_n(X_1, \ldots, X_n) = \sum_{\rho\in\mathcal{P}(n)} k_\rho(X_1, \ldots, X_n).$$

Example

There are 5 partitions of 3 elements:



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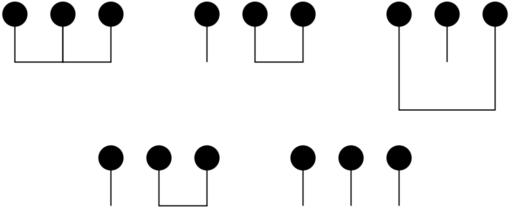

Thus

$$\mathbb{E}\left(XYZ\right) = a_3\left(X,Y,Z\right) = k_3\left(X,Y,Z\right) + k_1\left(X\right)k_2\left(Y,Z\right) + k_2\left(X,Z\right)k_1\left(Y\right) + k_2\left(X,Y\right)k_1\left(Z\right) + k_1\left(X\right)k_1\left(Y\right)k_1\left(Z\right).$$

The first four cumulants are:

$$k_1(X) = \mathbb{E}(X)$$

$$k_2(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$k_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(X)\mathbb{E}(YZ) - \mathbb{E}(XY)\mathbb{E}(Y) - \mathbb{E}(XY)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z)$$

$$\begin{aligned} k_4(X, Y, Z, W) = {} & \mathbb{E}(XYZW) - \mathbb{E}(X)\mathbb{E}(YZW) \\ & - \mathbb{E}(XZW)\mathbb{E}(Y) - \mathbb{E}(XYW)\mathbb{E}(Z) - \mathbb{E}(XYZ)\mathbb{E}(W) \\ & - \mathbb{E}(XY)\mathbb{E}(ZW) - \mathbb{E}(XZ)\mathbb{E}(YW) - \mathbb{E}(XW)\mathbb{E}(YZ) \\ & + 2\mathbb{E}(XY)\mathbb{E}(Z)\mathbb{E}(W) + 2\mathbb{E}(XZ)\mathbb{E}(Y)\mathbb{E}(W) \\ & + 2\mathbb{E}(XW)\mathbb{E}(Y)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(YZ)\mathbb{E}(W) \\ & + 2\mathbb{E}(X)\mathbb{E}(YW)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(ZW) \\ & - 6\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z)\mathbb{E}(W). \end{aligned}$$

Cumulants correspond to connected surfaces (asymptotically).

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For any cumulant, we have an Euler characteristic expansion:

$$\begin{gathered}\text{(sphere terms)}\, N^{-2r+2} + \text{(projective plane terms)}\, N^{-2r+1} + \\ \text{(torus and Klein bottle terms)}\, N^{-2r} + \\ \text{(connected sum of 3 projective planes terms)}\, N^{-2r-2} + \cdots .\end{gathered}$$

Let $A_1, \ldots, A_r$ be in the algebra generated by alternating ensembles of random matrices. If we expand out an expression of the form

$$\mathbb{E}\left(\mathrm{tr}\left(\left(A_1 - \mathbb{E}\left(\mathrm{tr}\left(A_1\right)\right)\right)\cdots\left(A_r - \mathbb{E}\left(\mathrm{tr}\left(A_r\right)\right)\right)\right)\right)$$

we get

$$\sum_{I \subseteq [r]} (-1)^{|I|} \prod_{i \in I} \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right) \mathbb{E}\left(\mathrm{tr}\left(\prod_{i \notin I} A_i\right)\right).$$

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For an $I \subseteq [r]$, the associated term is given by diagrams in which the A_i are disconnected for $i \in I$.

Expressions like this one can be interpreted in terms of the Principle of Inclusion and Exclusion.

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Since diagrams with connected A_i require crossings, these vanish asymptotically.

In order to find an appropriate definition of second-order freeness, we want to consider values of

$$\lim_{N\to\infty} k_2 \left(\mathrm{Tr}\left(\left(A_1 - \mathbb{E}\left(\mathrm{tr}\left(A_1\right)\right)\right)\cdots\left(A_p - \mathbb{E}\left(\mathrm{tr}\left(A_p\right)\right)\right)\right),\right.$$
$$\left.\mathrm{Tr}\left(\left(B_1 - \mathbb{E}\left(\mathrm{tr}\left(B_1\right)\right)\right)\cdots\left(B_q - \mathbb{E}\left(\mathrm{tr}\left(B_q\right)\right)\right)\right)\right).$$

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$$\left.\mathrm{Tr}\left(\left(B_1-\mathbb{E}\left(\mathrm{tr}\left(B_1\right)\right)\right)\cdots\left(B_q-\mathbb{E}\left(\mathrm{tr}\left(B_q\right)\right)\right)\right)\right).$$

We can apply the Principle of Inclusion and Exclusion to this expression as well, with the same interpretation.

Now the A_i can be connected to the B_i.

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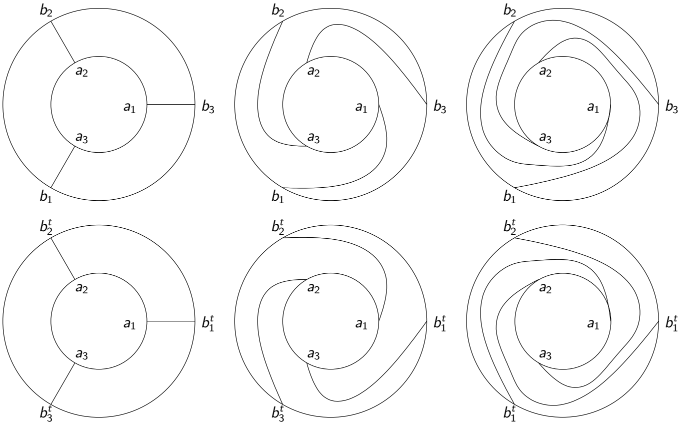
If $p = q$, then we must construct a "spoke diagram".

In the real case, unlike the complex case, we need to consider spoke diagrams with both relative orientations.

For $p = 3$, we must consider these six possible configurations:



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The contribution of such a spoke is

$$\mathbb{E}\left(\mathrm{tr}\left(A_i B_j\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_j\right)\right)$$

or

$$\mathbb{E}\left(\mathrm{tr}\left(A_i B_j\right)\right) - \mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_j^T\right)\right).$$

Definition

Random matrices $X_{c,N} : \Omega \to (\mathbb{C})$ are *asymptotically real second-order free* if, for cyclically alternating (or length 1) words in the colours v and w and for $A_{k,N}$ in the algebra generated by $X_{v(k),N}^{(\pm\varepsilon)}$ and $B_{k,N}$ in the algebra generated by $X_{w(k),N}^{(\pm\varepsilon)}$, the expression

$$\lim_{N\to\infty} k_2 \left(\mathrm{Tr}\left(\left(A_1 - \mathbb{E}\left(\mathrm{tr}\left(A_1\right)\right)\right)\cdots\left(A_p - \mathbb{E}\left(\mathrm{tr}\left(A_p\right)\right)\right)\right),\right.$$
$$\left.\mathrm{Tr}\left(\left(B_1 - \mathbb{E}\left(\mathrm{tr}\left(B_1\right)\right)\right)\cdots\left(B_q - \mathbb{E}\left(\mathrm{tr}\left(B_q\right)\right)\right)\right)\right)$$

is equal to 0 whenever $p \neq q$,

Definition (cont'd)

and equal to

$$\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\mathbb{E}\left(\mathrm{tr}\left(A_1 B_{k-i}\right)\right)-\mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k-i}\right)\right)\right)$$
$$+\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\mathbb{E}\left(\mathrm{tr}\left(A_i B_{k+i}^T\right)\right)-\mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k+i}^T\right)\right)\right)$$

whenever $p = q \geq 2$.

Definition

Subspaces $A_1, \ldots, A_n$ of a second-order probability space A with involution $A \mapsto A^t$ are *real second-order free* if they are free, and

$$\rho\left(a_1 \cdots a_p, b_1 \cdots b_q\right) = 0$$

whenever $p \neq q$ with $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and cyclically alternating (or consisting of a single factor), and

$$\rho\left(a_1 \cdots a_p, b_1, \cdots, b_p\right) = \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi\left(a_i b_{k-i}\right) + \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi\left(a_i b_{k+i}^t\right)$$

with $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ centred and cyclically alternating.