Limit Theorems for Spectral Statistics of Random Matrices

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Outline

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  - Gaussian Matrices: LLN, CLT
  - Wigner Matrices: LLN, CLT
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- CLT IS NOT VALID
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  - Matrix Elements of Functions of Wigner Matrices

Let $\{\xi_l\}_{l=1}^n$ be i.i.d. r.v's with the probability law F and $\varphi : \mathbb{R} \to \mathbb{R}$ and

$$\mathcal{N}_n[\varphi] = \sum_{l=1}^{n} \varphi(\xi_l)$$

be the linear statistic of $\{\xi_l\}_{l=1}^n$, corresponding to the test function φ.

- Law of Large Numbers (LLN): if $\mathbf{E}\{|\varphi(\xi_1)|\} < \infty$, then with probability 1

$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda)F(d\lambda).$$

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$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda)F(d\lambda).$$
- Central Limit Theorem (CLT): if $\mathbf{E}\{\varphi^2(\xi_1)\} < \infty$ then $n^{-1/2}(\mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\})$ converges in distribution to the Gaussian r.v. with mean zero and variance
$$v^2 = \lim_{n\to\infty} n^{-1}\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = \mathbf{Var}\{\varphi(\xi_1)\}, \ \mathbf{Var}\{\eta\} := \mathbf{E}\{\eta^2\} - \mathbf{E}^2\{\eta\}.$$

Note that for i.i.d. r.v.'s $\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = O(n), \ n \to \infty$

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- We are interested in the limiting laws of $\mathcal{S}_n$ and $\mathcal{N}_n$ as $n \to \infty$. (possibly after a certain additive and/or multiplicative normalization (recall LLN and CLT for i.i.d. r.v.'s).

Set $M_n = n^{-1/2} W_n, \ W_n = \{W_{jk}\}_{j,k=1}^{n}$

$$P_n(dW) = Z_n^{-1} e^{-\mathrm{Tr} W^2/2w^2} \prod_{1 \le j \le n} dW_{jj} \prod_{1 \le j \le k \le n} d\,\mathrm{Re}\, W_{jk}\, d\,\mathrm{Im}\, W_{jk}.$$

Since

$$\mathrm{Tr} W_n^2 = \sum_{1 \le j \le n} W_{jj}^2 + 2 \sum_{1 \le j \le k \le n} |W_{jk}|^2,$$

the above implies that $\{W_{jk}\}_{1 \le j \le k \le n}$ are independent Gaussian random variables such that

$$\mathbf{E}\{W_{jk}\} = \mathbf{E}\{W_{jk}^2\} = 0, \ \mathbf{E}\{|W_{jk}|^2\} = w^2(1 + \delta_{jk})/2.$$

Gaussian Unitary Ensemble (GUE)

Warning

Take the GOE for $n = 2$, i.e.,

$$Z_2^{-1} e^{-\mathrm{Tr} M^2/2w^2} dM_{11} dM_{22} d\,\mathrm{Re}\, M_{12} d\,\mathrm{Im}\, M_{12}.$$

and find the joint distribution of (λ_1, λ_2):

$$Q_2^{-1} e^{-(\lambda_1^2+\lambda_2^2)/2w^2} |\lambda_1 - \lambda_2|^2 d\lambda_1 d\lambda_2,$$

since

$$\lambda_{1,2} = \frac{(M_{11} + M_{22}) \mp \sqrt{(M_{11} - M_{22})^2 + |M_{12}|^2}}{2}.$$

Eigenvalues are strongly dependent even if the matrix elements are not!

Theorem

Let M_n be the GUE matrix and $\mathcal{N}_n[\varphi]$ be a linear eigenvalue statistics of its eigenvalues. Then we have for any bounded and continuous $\varphi : R \to C$ with probability 1:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \varphi\left(\lambda_l^{(n)}\right) = \int \varphi(\lambda) N_{scl}(d\lambda),$$

where the measure

$$N_{sc}(d\lambda) = \rho_{sc}(\lambda)d\lambda,\ \rho_{sc}(\lambda) = (2\pi w^2)^{-1}\sqrt{4w^2 - \lambda^2}\mathbf{1}_{|\lambda|\le 2w}$$

is known as the Wigner or the semicircle law.

Wigner 52 and many others.

Take the Gaussian matrix with non-zero mean: $H_n = H_{0,n} + M_n$, assume that the Normalizing Counting Measure (NCM) $N_{0,n}$ of eigenvalues of $H_{0,n}$ (which can be random but independent of M_n) converges weakly to N_0. Then the NCM N_n of H_n converges weakly with probability 1 to a non-random limit N (hence any linear eigenvalue statistics with bounded and continuous test function does) and if

$$f(z) = \int \frac{N(d\lambda)}{\lambda - z}, \ \Im z \neq 0;$$

is its *Stieltjes transform* and f_0 is that for N_0, then $f(z) = f_0(z + w^2 f(z))$ and the equation is uniquely solvable in the class of functions analytic in $\mathbb{C} \setminus \mathbb{R}$ and such that $\mathrm{Im}\, f(z)\, \mathrm{Im}\, z > 0, \ \mathrm{Im}\, z \neq 0$ (*Nevanlinna class* $\mathcal{N}$) and $f(z) = -z^{-1} + o(z^{-1}), \ z \to \infty$.
This is known as the deformed semicircle law *P. 72*.

Theorem

Let M_n be the GUE matrix, $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a polynomially bounded derivative. Then $\mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$!!! converges in distribution to the Gaussian random variable with zero mean and the variance

$$V_{GOE}[\varphi] = \frac{1}{4\pi^2}\int_{-2w}^{2w}\int_{-2w}^{2w}\left(\frac{\varphi(\lambda_1)-\varphi(\lambda_2)}{\lambda_1-\lambda_2}\right)^2 \times \frac{4w^2-\lambda_1\lambda_2}{\sqrt{4w^2-\lambda_1^2}\sqrt{4w^2-\lambda_2^2}}d\lambda_1 d\lambda_2.$$

Khorunzhy, Khoruzhenko, P. 96; Johansson 98; Guionnet et al 2000's and others

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = O(1),\ n \to \infty \quad \text{A PUZZLE ?!}$$

LLN $\Longrightarrow \lambda_l^{(n)} = O(1)$, $n \to \infty$, moreover, asymptotically are in $[-2w, 2w]$ with p.1, i.e.,

$$\mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\} = \sum_{l=1}^{n} O(1),$$

thus the CLT could result from the strong cancelations of terms.
Examples: recall that $M_n = n^{-1/2} W_n$ and consider:

(i) $\varphi(\lambda) = \lambda$, where $\sum_{l=1}^{n} \lambda_l^{(n)} = \mathrm{Tr} M = n^{-1/2} \sum_{j=1}^{n} W_{jj}$

is Gaussian by definition;

(ii) $\varphi(\lambda) = \lambda^2$, where $\sum_{j,k=1}^{n} (\lambda_l^{(n)})^2 = \mathrm{Tr} M^2 = n^{-1} \sum_{j,k=1}^{n} |W_{jk}|^2$

is asymptotically Gaussian by standard CLT.

The "same" for sufficiently regular φ.

Take hermitian matrices A_n and B_n having limiting NCM's N_A and N_B and the Haar distributed unitary matrix U_n and write

$$H_n = A_n + U_n B_n U_n^* \tag{1}$$

Analogous real symmetric matrices with the orthogonal Haar distributed matrix instead of the unitary.

The model is known since long time but became popular after *Voiculescu* works of the 80s-90s and free probability.

Let $\{\beta_l\}_{l=1}^n$ and $\{b_l\}_{l=1}^n$ be the eigenvalues and eigenvectors of B_n. Then

$$H_n = A_n + \sum_{l=1}^{n} \beta_l P_{q_l},$$

where $\{P_{q_l}\}_{l=1}^n$ are the orthogonal projections on the random vectors $\{q_l\}_{l=1}^n$ uniformly distributed over the the unit sphere in $\mathbb{C}^n$, modulo their pairwise orthogonality. Removing this restriction, we obtain the i.i.d. random vectors uniformly distributed over the the unit sphere in $\mathbb{C}^n$. This case was considered by *Marchenko, P. 67*.

The cases of the deformed GUE and the deformed Laguerre Ensemble $M_n = X * X,\ X = \{X_j k\}_{j,k=1}^n$ are also the particular cases of (1) with certain random B_n.

It can be proved then that the NCM of $H_n$converges weakly with probability 1 to a non-random limit Nwhose Stiletjes transform solves the system

$$\begin{cases} f_{A_1+A_2}(z) & = & f_{A_1}(h_{A_1}(z)), \\ f_{A_1+A_2}(z) & = & f_{A_2}(h_{A_2}(z)), \\ f^{-1}_{A_1+A_2}(z) & = & z - h_{A_1}(z) - h_{A_2}(z), \end{cases}$$

where f_A and f_B the Stieltjes transforms of N_A and N_B and $h_{A_{1,2}}(z)$ analytic in $\mathbb{C}\backslash\mathbb{R}$, $h_{A_{1,2}}(z) = z + o(z),\ z \to \infty$.
Voiculescu 80s; Speicher 90s; P., Vasilchuk 00, 07 with a long and then a short and transparent one, based on a version of the Poincaré inequality for classical groups.

Consider $H_n = A_n + U_n B_n U_n^*$ and assume

$$\sup_n \max \left\{ \int |\lambda|^4 N_{n,A}(d\lambda), \int |\lambda|^4 N_{n,B}(d\lambda) \right\} < \infty.$$

Then $\overset{\circ}{\gamma}_n(z) = \gamma_n(z) - \mathbf{E}\{\gamma_n(z)\}$, where $\gamma_n(z) = \mathrm{Tr}\,(H - z)^{-1}$and $z \in \mathbb{C}\backslash\mathbb{R}$, converges in distribution to the complex Gaussian random variable $\gamma(z)$ with zero means and the covariances

$$\begin{aligned}
\mathbf{Var}\{\Re\gamma(z)\} &= 2^{-1}\Re(S(z,z) + S(z,\overline{z})), \\
\mathbf{Var}\{\Im\gamma(z)\} &= -2^{-1}\Re(S(z,z) - S(z,\overline{z})), \\
\mathbf{Cov}\{\Re\gamma(z), \Im\gamma(z)\} &= 2^{-1}\Im(S(z,z) - S(z,\overline{z})),
\end{aligned}$$

in which

$$S(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \log \frac{(h_A(z_1) - h_A(z_2))(h_B(z_1) - h_B(z_2))}{(z_1 - z_2)(f^{-1}(z_1) - f^{-1}(z_2))}$$

is limit the of **Cov**$\{\gamma_n(z_1), \gamma_n(z_2)\}$ as $n \longrightarrow \infty$, and $h_{A,B}$ and f are as above.

Since

$$\gamma_n(z) = \sum_{l=1}^{n} \left(\lambda_l^{(n)} - z\right)^{-1}$$

and its variance is $O(1)$ but not $O(n)$, we have the same phenomenon of strong cancellation as for the Gaussian Ensembles.

Related results by *Speicher et al.*

$$M_n = n^{-1/2} W_n, \; W_n = \{W_{jk}\}_{j,k=1}^{n}$$

with $W_{jk} = W_{kj} \in \mathbb{R}$, $1 \le j \le k \le n$ independent and

$$\mathbf{E}\{W_{jk}\} = 0, \quad \mathbf{E}\{W_{jk}^2\} = (1+\delta_{jk})w^2,$$

i.e. the two first moments of the entries coincide with those of the GOE or

$$\mathbf{P}(dW_n) = \prod_{1 \le j \le k \le n} F_{jk}(dW_{jk}),$$

where F_{jk} has above moments. The GOE corresponds to

$$F_{jk}(dW) = \frac{1}{(2\pi\sigma_{jk}^2)^{1/2}} e^{-W^2/2\sigma_{jk}^2} dW, \quad \sigma_{jk}^2 = (1+\delta_{jk})w^2.$$

(i) Law of Large Numbers

Theorem

Let $M_n = n^{-1/2} W_n$ be the Wigner matrix and N_n be the Normalized Counting Measure of its eigenvalues. We have
(i) if $\sup_{j,k} \mathbf{E}\{|W_{jk}|^{2+\delta}\} < \infty$ for some $\delta > 0$, then the semicrcle law is valid with probability 1:

P 72, Girko 75

The LLN is the same as for Gaussian matrices (macroscopic universality) *P 72, Girko 75*;

(ii) if

$$w_6 := \sup_{j,k} \mathbf{E}\{(W_{jk}^6\} < \infty,$$

and $(1 + |t|^5)|\widehat{\varphi}(t)| \in L^1(\mathbb{R},$ then $\mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$ obeys (!?) the CLT with the variance

Theorem

Let U_n be a $n \times n$ unitary random matrix, whose probability law is the normalized Haar measure on $U(n)$, and A_n be a $n \times n$ matrix such that

$$\lim_{n\to\infty} n^{-1}\mathrm{Tr} A_n^* A_n = 1.$$

Then $\mathrm{Tr} U_n A_n$ converges in distribution to the standard complex Gaussian variable: $\gamma = \gamma_1 + i\gamma_2$, $\mathbf{E}\{\gamma_1\} = \mathbf{E}\{\gamma_1\} = 0$, $\mathbf{E}\{\gamma_1^2\} = \mathbf{E}\{\gamma_2^2\} = 1/2$.

E. Borel 1905 ($A_n = \{\delta_{j1}\delta_{k1}\}_{j,k=1}^n$, $\mathrm{Tr} U_n A_n = O_{11}$), Diaconis et al 2003; Collins, Stolz 2006; P. 2007.

Not linear eigenvalue statistics

$$\mathrm{Tr}\varphi(M_n) = \sum_{j=1}^{n} \varphi_{jj}(M_n) = \sum_{l=1}^{n} \varphi(\lambda_l^{(n)})$$

but a (simple) spectral statistic

$$\varphi_{jj}(M_n) = \sum_{l=1}^{n} \varphi(\lambda_l^{(n)})|\psi_l^{(n)}|^2.$$

Since

$$\sum_{l=1}^{n} |\psi_l^{(n)}|^2 = 1$$

$|\psi_l^{(n)}|^2 \simeq 1/n$, it is reasonable to believe that $\varphi_{jj}(M_n)$ is to be asymptotically "analogous" to $1/n\times$ linear eigenvalue statistics.

Theorem

Let M_n be the GOE matrix and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous. We have for any $j_n \to \infty$ as $n \to \infty$

- $\mathbf{E}\{\varphi_{j_n j_n}(M_n)\}) = \mathbf{E}\{n^{-1}\mathrm{Tr}\varphi(M_n)\}$ *and with p. 1*

$$\lim_{n\to\infty} \varphi_{jj}(M) = \int \varphi(\lambda) N_{scl}(d\lambda);$$

- $\lim_{n\to\infty} n\mathbf{Var}\{\varphi_{j_n j_n}(M_n)\} = V^{(d)}_{GOE}[\varphi]$, *where*

$$V^{(d)}_{GOE}[\varphi] = \frac{1}{2}\int\int |\varphi(\lambda_1) - \varphi(\lambda_2)|^2 N_{scl}(d\lambda_1) N_{scl}(d\lambda_2);$$

- $n^{1/2}(\varphi_{j_n j_n}(M_n) - \mathbf{E}\{\varphi_{jj}(M_n)\})$ *obey the CLT with variance* $V^{(d)}_{GOE}[\varphi]$

Lytova, P. 09 LLN as for the traces, the variance is $O(n^{-1})$ and the CLT, One distinguish now the cases

$$H_n = A_n + U_n^* B U_n, \tag{2}$$

and

$$\tilde{H} = V_n^* A_n V_n + U_n^* B_n U_n. \tag{3}$$

which have the same results for the eigenvalue statistics (by the shift invariance of the Haar measure and unitary invariance of eigenvalues).
Assume that $\sup_n ||A_n||, ||B_n|| < \infty$ and denote $G_n(z)$ the resolvent one of above random matrices.
Then we have in both cases the same LLN, i.e., the convergence with probability 1 of $(G_n(z))_{kk}$ to the same limit as $n^{-1}\mathrm{Tr}\, G_n(z)$, i.e., for the solution of the above system.
However, this is not the case for the fluctuations. Here we have

$$\mathbf{Cov}\{(G_n(z_1))_{kk}, (G_n(z_2))_{kk}\} = \frac{1}{n} T_n(z_1, z_2) + r_n(z_1, z_2), \; z_{1,2} \in \mathbb{C} \backslash \mathbb{R}$$

where in the case of (2)

and in the case of (3)

$$T_n(z_1, z_2) = \frac{\delta f}{\delta z} - f(z_1) f(z_2) \qquad (4)$$

in which

$$\begin{aligned} \delta z &= z_1 - z_2, \ \delta f = f(z_1) - f(z_2), \\ \delta h_B &= h_B(z_1) - h_B(z_2), \ G_A(z) = (A - zI)^{-1} \end{aligned}$$

and the remainder $r_n(z_1, z_2)$ admits the bound

$$|r_n(z_1, z_2)| \leq C/n^{3/2},$$

where C is independent of n and is finite if $\min\{|\Im z_1|, |\Im z_2|\} > 0$. Moreover, the CLT with this variance is valid for sufficiently wide class of test functions.
Thus the situation is similar to that in the probability theory.

Theorem

Let $M_n = n^{-1/2}W_n,\ W_n = \{W_{jk}\}_{j,k=1}^n$ *be real symmetric Wigner matrix such that the 3rd and 4th moments of* W_{jk} *do not depend on* j, k*. Consider* $\varphi_{jj}(M_n)$*. where* $(1+|t|)^3\widehat{\varphi}(t) \in L^1(\mathbb{R})$*. Then we have for any* $j_n \to \infty$ *as* $n \to \infty$ *:*

- *with probability 1*

$$V_d^W[\varphi] := \lim_{n\to\infty} n\mathbf{Var}\{\varphi_{j_nj_n}(M)\}$$
$$= V_d^{GOE}[\varphi] + \frac{\kappa_4}{w^8}\Big(\int_{-2w}^{2w}\varphi(\mu)(w^2-\mu^2)\rho_{sc}(\mu)d\mu\Big)^2,$$

where ρ_{sc} *is the density of the semicircle law.*

Thus, the variance of $\varphi_{j_nj_n}(M)$ is $O(1/n)$ as for the GOE (although has an additional term). Is the CLT the case?

Theorem

Consider $M_n = n^{-1/2} W_n,\ W_n = \{W_{jk}\}_{j,k=1}^n,\ W_{jk} = (1+\delta_{jk})^{1/2} V_{jk}$, *where* $V_{jk},\ 1 \le j \le k \le n$ *are i.i.d. Assume that the logarithm of the characteristic function* $\mathbf{E}\{e^{ixV_{11}}\}$ *is entire. Then* $\sqrt{n}\varphi^{\circ}_{j_n j_n}(M)$ *converges in distribution as* $n \to \infty$ *to the random variable* ξ, *such that*

$$\mathbf{E}\{e^{ix\xi}\} = \exp\left\{-(V_d^W x^2 + w^2 (x^*)^2)/2\right\} \mathbf{E}\{e^{ix^* V_{11}}\}$$

where

$$x^* = \frac{x}{w^2} \int_{-2w}^{2w} \varphi(\mu)\mu\rho_{sc}(\mu)d\mu.$$

Lytova, P. 10. "Almost" individual

Analogous result for infinitely divisible $\{V_{jk}\}$'s and $\varphi \in C^2$ *Lytova 10.*

Eigenvectors are not too similar to those of GOE

(i) $\varphi(\lambda) = \lambda$:

$$n^{1/2}\varphi_{jj}(M_n) = W_{jj} = \begin{cases} \text{Gaussian}, & GOE, \\ \text{any}, & Wigner \neq GOE. \end{cases}$$

(ii) $\varphi(\lambda) = \lambda^2$:

$$n^{1/2}\varphi_{jj}(M_n) = n^{-1/2}\sum_{k=1}^{n} W_{jk}^2$$

is asymptotically Gaussian by standard CLT.

The "same" for sufficiently regular φ's, since the "renormalized" argument x^* of $E\{e^{ixV_{11}}\}$ is

$$x^* = \begin{cases} \text{proportional } x & \varphi \text{ is odd}, \\ 0, & \varphi \text{ is even}. \end{cases}$$

$$Z_n^{-1} \exp\{-\mathrm{Tr}\, V(M_n)\} dM_n$$

$$dM_n = \prod_{j=1}^{n} dM_{jj} \prod_{1 \le j < k \le n} d\Re M_{jk} d\Im M_{jk},$$

$V : \mathbb{R} \to \mathbb{R}_+$ is a continuous function (potential), and

$$\exists\, \varepsilon > 0,\ L < \infty \quad V(\lambda) \ge (2+\varepsilon) \log(1+|\lambda|) > 0,\ |\lambda| \ge L$$

$V = \lambda^2/2$ corresponds to the Gaussian Unitary Ensemble (GUE). In fact,

$$(\mathit{Wigner\ Matrices}) \bigcap (\mathit{Matrix\ Models}) = (\mathit{Gaussian\ Matrices})$$

Strongly dependent entries, e.g. for $V = \lambda^4$

For any non-negative measure m of unit mass on $\mathbb{R}$ define (*Gauss*)

$$\mathcal{E}[m] = \int V(\lambda)m(d\lambda) - \int\int \log|\lambda-\mu| m(d\lambda)m(d\mu),$$

and let N be a unique minimizer of $\mathcal{E}$: $\inf_m \mathcal{E}[m] = \mathcal{E}(N)$. Then for $V' \in Lip_{loc}1$ with probability 1 in weak sense

$$\lim_{n\to\infty} n^{-1}\mathcal{N}_n = N, \; N(d\lambda) = \rho(\lambda)d\lambda,$$

Wigner 52; Brezin et al 79; A. Boutet de Monvel, P., Shcherbina 95; Deift et al 98; Johansson, 98
If V is convex, then $\mathrm{supp} N = [a,b]$ and if V is real analytic, then

$$\mathrm{supp} N = \bigcup_{l=1}^{q} [a_l, b_l], \; 1 \le q < \infty.$$

- $O(1)$ bound for *Lip*$_1$ test functions *P., Shcherbina 97*

- $O(1)$ bound for *Lip_1* test functions *P., Shcherbina 97*
- Asymptotic form

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq \mathcal{V}(n\beta),$$

$$\mathcal{V} : \mathbb{T}^{q-1} \to \mathbb{R}_+, \ \beta_l = N([a_l, \infty)), \ l = 2, ..., q$$

Note that $\mathcal{V}$ is the quadratic functional of φ.
For $q \geq 2$, $\mathbf{Var}\{\mathcal{N}_n[\varphi]\}$ is asymptotically quasiperiodic in n , thus no limit as $n \to \infty$; its sublimits are indexed by $x \in \mathbb{H}^{q-1} \in \mathbb{T}^{q-1}$, hence the family of CLT's, indexed by $x \in \mathbb{H}^{q-1}$ (generalized CLT).
P. 06

$$Z_n[\varphi] := \mathbf{E}_V \left\{ e^{-\mathcal{N}_n^\circ[\overline{\varphi}]} \right\} " \longrightarrow " e^{\Phi[\varphi]}, \quad n \to \infty$$

with

$$\Phi[\varphi] = \int_0^1 (1-s)\mathcal{V}(x + s\alpha[\varphi])ds$$

$$\alpha_l[\varphi] = \int \frac{\delta\beta_l}{\delta V(\lambda)} \varphi(\lambda) d\lambda, \; l = 1, ..., q-1.$$

Z_n has no limit as $n \to \infty$ in general. The logarithms of its sublimits (indexed by $\mathbb{H}^{q-1}$) are not in general quadratic in φ (coinciding asymptotically with " lim"$_{n\to\infty}\mathbf{Var}_V \{\mathcal{N}_n[\varphi]\}/2$), hence no (generalized) CLT in general. *P. 06*

The "explanation" used for the Wigner matrices does not apply since the entries of M_n are strongly dependent now.

$\varphi = t\lambda,\ t \in \mathbb{R}$, i.e., $\mathcal{N}_n = t\left(\lambda_1^{(n)} + \ldots + \lambda_n^{(n)}\right)$,

$$V(\lambda) = \frac{\lambda^4}{4} - c\frac{\lambda^2}{2},\quad c > \sqrt{2},\ \mathrm{supp}\ N = [-b, -a] \cup [a, b]$$

hence $\beta_1 = 1/2$ and $\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \simeq t^2(b^2 + a^2 - 2(-1)^n ab)/4$
2-periodic.
However $\alpha_1 = \alpha[\lambda]|_{\varphi=\lambda} = a\big/K(a/b)$, where K is the complete elliptic integral of first kind and, is generically irrational and

$$\begin{aligned}\Phi &= \frac{d_0 t^2}{2} + A(x + \alpha_1 t) - A(x) - \alpha_1 t A'(x),\ \{x = n/2\}\\ d_0 &= \frac{a^2 + b^2}{4} \neq \text{"}\lim\text{"}_{n\to\infty} \mathbf{Var}\{\mathcal{N}_n\},\\ A(x) &= \sum_{m \in \mathbb{Z}\setminus\{0\}} d_m (2\pi i \alpha_1 m)^{-2} e^{2\pi i m x}\end{aligned}$$

Φ is quasiperiodic (and not quadratic!) in t.

Theorem

Let $M_n = n^{-1/2}W_n, \; W_n = \{W_{jk}\}_{j,k=1}^n$ *be real symmetric Wigner matrix such that the 3rd and 4th moments of* W_{jk} *do not depend on* j, k*. Consider* $\varphi_{jj}(M_n)$*. where* $(1+|t|)^3\widehat{\varphi}(t) \in L^1(\mathbb{R})$*. Then we have for any* $j_n \to \infty$ *as* $n \to \infty$ *:*

- *with probability 1*

$$V_d^W[\varphi] := \lim_{n\to\infty} n\mathbf{Var}\{\varphi_{j_nj_n}(M)\}$$
$$= V_d^{GOE}[\varphi] + \frac{\kappa_4}{w^8}\Big(\int_{-2w}^{2w}\varphi(\mu)(w^2-\mu^2)\rho_{sc}(\mu)d\mu\Big)^2,$$

where ρ_{sc} *is the density of the semicircle law.*

Thus, the variance of $\varphi_{j_nj_n}(M)$ is $O(1/n)$ as for the GOE (although has an additional term). Is the CLT the case?

Theorem

Consider $M_n = n^{-1/2}W_n,\ W_n = \{W_{jk}\}_{j,k=1}^n,\ W_{jk} = (1+\delta_{jk})^{1/2}V_{jk}$, *where* $V_{jk},\ 1 \le j \le k \le n$ *are i.i.d. Assume that the logarithm of the characteristic function* $\mathbf{E}\{e^{ixV_{11}}\}$ *is entire. Then* $\sqrt{n}\varphi^{\circ}_{j_nj_n}(M)$ *converges in distribution as* $n \to \infty$ *to the random variable* ξ*, such that*

$$\mathbf{E}\{e^{ix\xi}\} = \exp\left\{-(V_d^W x^2 + w^2(x^*)^2)/2\right\}\mathbf{E}\{e^{ix^*V_{11}}\}$$

where

$$x^* = \frac{x}{w^2}\int_{-2w}^{2w}\varphi(\mu)\mu\rho_{sc}(\mu)d\mu.$$

Lytova, P. 10. "Almost" individual

Analogous result for infinitely divisible $\{V_{jk}\}$'s and $\varphi \in C^2$ *Lytova 10.*

Eigenvectors are not too similar to those of GOE

Tools:

- Gaussian differentiation formula (integration by parts):

$$\mathbf{E}\{\xi_l \Phi(\xi)\} = \mathbf{E}\{\xi_l^2\}\mathbf{E}\{\Phi_l'(\xi)\}, \ l = 1, ..., p;$$

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- Poincaré(Nash-Chernoff) inequality:

$$\mathbf{Var}\{\Phi\} \le \sum_{l=1}^{p} \mathbf{E}\{\xi_l^2\}\mathbf{E}\left\{|\Phi_l'|^2\right\},$$

valid for a collection $\{\xi_l\}_{l=1}^p$ of independent Gaussian random variables.

Bound for the Variance of Linear Eigenvalue Statistics

Observe that for the GOE $\{M_{jk}\}_{1\leq j\leq k\leq n}$ are independent Gaussian,

$$\mathbf{Var}\{M_{jk}\} = w^2(1+\delta_{jk})/n$$

and

$$\frac{\partial \mathrm{Tr}\varphi(M)}{\partial M_{jk}} = \frac{w^2}{n}(1+\delta_{jk})\varphi'_{jk}(M).$$

Then the Poincaré yields:

$$\begin{aligned}\mathbf{Var}\{\mathrm{Tr}\varphi(M)\} &\leq 2w^2\mathbf{E}\{n^{-1}\mathrm{Tr}\varphi'(M)(\varphi'(M))^*\} \\ &\leq 2w^2 \sup_{\lambda\in\mathbb{R}} |\varphi'(M)|^2!!!\end{aligned}$$

Semicircle Law

Consider the Stieltjes transform of N_n

$$g_n(z) = \int \frac{N_n(d\lambda)\}}{\lambda - z}, \ \Im z \neq 0.$$

By spectral theorem $g_n(z) = n^{-1}\mathrm{Tr}\, G(z)$, by resolvent identity

$$f_n(z) := \mathbf{E}\{g_n(z)\} = z^{-1} + (zn)^{-1} \sum_{j,k=1}^{n} \mathbf{E}\{M_{jk} G_{kj}(z)\},$$

by dif. formula $f_n(z) = z^{-1} + z^{-1}\mathbf{E}\{g_n^2(z)\} + O(1/n)$, and by the bound for the variance

$$f(z) = z^{-1} + w^2 z^{-1} f^2(z)$$

for $\lim_{n\to\infty} f_n = f$ uniformly on compacts of $\mathbb{C}\setminus\mathbb{R}$. Then $\mathrm{Im}\, f(z)\, \mathrm{Im}\, z > 0$ and inversion formula give semicircle law for $\lim_{n\to\infty} \mathbf{E}\{N_n\}$.
Bose, Chatterjee 04; P. 05

Wigner Matrices

- Martingale bounds $\mathbf{E}\{|N_n^\circ(\Delta)|^4\} = O(n^{-2})$ instead of Poincaré *Girko 70s*

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If $\mathbf{E}\{|\xi|^{p+2}\} < \infty$, $p \in \mathbb{N}$, $\Phi : \mathbb{R} \to \mathbb{C}$ of C^{p+1} with bounded derivatives, then

$$\begin{aligned} \mathbf{E}\{\xi\Phi(\xi)\} &= \kappa_2 \mathbf{E}\{\Phi'(\xi)\} + \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\xi)\} + \varepsilon_p, \\ |\varepsilon_p| &\leq C_p \mathbf{E}\{|\xi|^{p+2}\} \sup_{t\in\mathbb{R}} |\Phi^{(p+1)}(t)|, \end{aligned}$$

where $\{\kappa_l\}_{l=1}^{\infty}$ are the cumulants of W_{12}. Note that the $l = 1$ term is "Gaussian".

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  where $\{\kappa_l\}_{l=1}^{\infty}$ are the cumulants of W_{12}. Note that the $l = 1$ term is "Gaussian".
- "Interpolation trick" *P. 00*: use the product space of the Wigner M_n and the GOE $\widehat{M}_n$ with the same first and second moments and set

$$M_n(s) = s^{1/2}M_n + (1-s)^{1/2}\widehat{M}_n, \quad 0 \leq s \leq 1,$$

Matrix Models

- Determinantal formulas for marginals of joint probability density:

$$p_{n,l}(\lambda_1, ..., \lambda_l) := \int_{\mathbb{R}^{n-l}} p_{n,l}(\lambda_1, ..., \lambda_l, \lambda_{l+1} ... \lambda_n) d\lambda_{l+1} ... d\lambda_n$$
$$= [n(n-1)...(n-l+1)]^{-1} \det\{K_n(\lambda_j, \lambda_k)\}_{j,k=1}^{l}$$

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- Asymptotics of $p_n^{(n)}$ and $p_{n-1}^{(n)}$ *Deift et al 97 - 99*