Non-commutative functions and some of their applications in free probability

Mihai Popa

Non-commutative probability spaces:

$(\mathcal{A}, \varphi)$, where

$\mathcal{A}$= unital, (C*-)($*$-)algebra

φ = normalized, (positive) $\mathbb{C}$-linear functional on $\mathcal{A}$

Independence relations ($\mathcal{A}_1, \mathcal{A}_2$=subalgebras of $\mathcal{A}$)

- Classical independence: $\mathcal{A}$=commutative

  $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$, $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$

- **Free** independence: $\varphi(a_1 a_2 \cdots a_n) = 0$ whenever $a_i \in \mathcal{A}_{k(i)}$ with $\varphi(a_k) = 0$ and $a_k \in \mathcal{A}_{\varepsilon(k)}$ with $k(i) \neq k(i+1)$ and $\varphi(a_k) = 0$.

- **Boolean** independence: $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$ whenever $a_j \in \mathcal{A}_{k(j)}$ with $k(i) \neq k(i+1)$.
- **Monotone** independence: $\varphi(\alpha x_1 \cdot y \cdot x_2 \beta) = \varphi(\alpha x_1 \cdot \varphi(y) \cdot x_2 \beta)$ whenever $x_k \in \mathcal{A}_1, y \in \mathcal{A}_2, \alpha, \beta \in$ alg$(\mathcal{A}_1, \mathcal{A}_2)$

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$X \in \mathcal{A}$, selfadjoint, determines a real measure μ_X via

$$\int t^n d\mu_X(t) = \varphi(X^n).$$

Limit distributions:

- Central Limit Theorems:
  - classical independence - Gaussian Distribution,
  - free independence - semicircular (Wiegner) Law (D-V. Voiculescu)
  - boolean independence - Bernoulli law (R. Speicher)
  - monotone independence - Arcsine law (N. Muraki)
- most general limits (<*infinitesimal arrays*) – infinitely divisible distributions

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Infinite divisibility wrt classical (additive) convolution: back to Kolmogorov, Hincin, P. Levy

Levy-Hincin formula:

$$\mathcal{F}\nu(t) = exp\left[i\alpha t + \int (e^{itx} - 1 - itx)\frac{x^2+1}{x^2}d\rho(x)\right]$$

Infinite divisibility wrt free additive convolution: H. Bercovici, D-V. Voiculescu ('92, '93), H. Bercovici, V. Pata ('95, '99)

Free Levy-Hincin formula:

$\Phi_\nu(z) = \gamma + \int \frac{1+tz}{z-t}d\sigma(t)$ for $\Phi_\nu(z) = \left(\frac{1}{G_\mu(z)}\right)^{\langle -1\rangle} - z$

or, equivalently

$\frac{1}{z}R_\nu(z) = \alpha + \int_{\mathbb{R}} \frac{z}{1-tz}d\rho(t)$ where $G_{\mu_X} \circ \left[\frac{1}{z}R_X(z) + \frac{1}{z}\right] = z$

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Operator-valued setting:

$\mathcal{B}$ = unital C*-algebra,

$\mathcal{B} \subset \mathcal{A}$ unital inclusion of $*$(C*)-algebras

$\phi : \mathcal{A} \longrightarrow \mathcal{B}$ positive conditional expectation

Example: $(\mathcal{A}, \varphi)$ =nc probability space

$S_N = [c_{i,j}]_{i,j=1}^N \in M_N(\mathbb{C}) \otimes \mathcal{A}$, independent, centered (semi)circulars, with $\varphi(c_{i,j}\overline{c_{i,j}}) = \frac{\sigma_{i,j}}{N}$

$A_N = [\alpha_{i,j}]_{i,j=1}^N \in M_N(\mathbb{C})$

Generally, S_N is **not** semicircular, and S_N and A_N are generally **not** free wrt to $tr \otimes \phi$, **but** wrt

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$$\phi(Xb_1Xb_2X) \neq \phi(X^3)b_1b_2$$

moments = multilinear maps

$$\mathcal{B}^n \ni (b_1, \ldots, b_n) \mapsto \phi(Xb_1 \cdots Xb_nX)$$

"Distributions"

$\mathcal{B}\langle\mathcal{X}\rangle$=the $*$-algebra of non-commutative polynomials in the self-adjoint variable $\mathcal{X}$ and with coefficients in $\mathcal{B}$

$\Sigma_{\mathcal{B}} = \{\nu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$: positive conditional expectation$\}$

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-Central Limit Laws: $b \mapsto \eta(b) =$ cp map

- **op-valued Bernoulli law:**

$$\mathrm{Ber}(\mathcal{X}b_1\cdots b_n\mathcal{X}) = \begin{cases} 0, & n = \text{even} \\ \eta(b_1)b_2\eta(b_3)\cdots b_{n-1}\eta(b_n) & n = \text{odd} \end{cases}$$

- **op-valued semicircular law** (D-V. Voiculescu, '95, R. Speicher '97)

$$\begin{aligned}\mathfrak{s}(\mathcal{X}b_1\mathcal{X}b_2\cdots\mathcal{X}b_5\mathcal{X}) =& \eta(b_1)b_2\eta(b_3)b_4\eta(b_5) + \eta(b_1)b_2\eta(b_3\eta(b_4)b_5) + \\ & \eta(b_1\eta(b_2)b_3)b_4\eta(b_5) + \eta(b_1\eta(b_2)b_3\eta(b_4)b_5) + \eta(b_1\eta(b_2\eta(b_3)b_4)b_5)\end{aligned}$$

easy using non-crossing pair partitions

- **op-valued arcsine law** (MP,'08)

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More general:

$\mathcal{B}$=unital C*-algebra
$\mathcal{A}$=unital $*$-algebra, $\mathcal{B} \subset \mathcal{A}$ unital inclusion of $*$-algebras
$\mathcal{D} = C^*$-algebra, $\mathcal{B} \subset \mathcal{D}$ inclusion of C$*$-algebras
$\theta : \mathcal{A} \longrightarrow \mathcal{D}$ c. p. $\mathcal{B}$-bimodule map

$\Sigma_{\mathcal{B}:\mathcal{D}} = \{\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$: unital $\mathcal{B}$-bimodule maps, c.p. condition$\}$.
C.p. condition: $f_1, \dots, f_N \in \mathcal{B}\langle\mathcal{X}\rangle$, then $\left[\nu(f_j^* \cdot f_i\right]_{i,j=1}^{N} \geq 0$ in $M_N(\mathcal{D})$

-Central Limit Laws:

- op-valued semicircular law (Voiculescu, '95)
- op-valued arcsine law (MP,'08)
- op-valued Bernoulli law

We need some appropriate analytic machinery for this framework, at least some good analogue for the Cauchy transform.

No literature about infinite divisibility in $\Sigma_{\mathcal{B}}$ except R. Speicher, Mem. AMS '98.

-Central Limit Laws:

- op-valued semicircular law (Voiculescu, '95)
- op-valued arcsine law (MP,'08)
- op-valued Bernoulli law

We need some appropriate analytic machinery for this framework, at least some good analogue for the Cauchy transform.

No literature about infinite divisibility in $\Sigma_{\mathcal{B}}$ except R. Speicher,
Mem. AMS '98.

Non-commutative functions

- J. L. Taylor (Adv. in Math. '72)
- D-V. Voiculescu (Asterisque '95, also 2008, '09) (∼fully matricial functions)
- V. Vinnikov, D.S. Kaliuzhnyi-Verbovetskyi, M. P., S. Belinschi (2009, '10)

Good analogue of the Cauchy transform, Taylor expansions, differential calculus

$\mathcal{V}$= vector space over $\mathbb{C}$;

- the *non-commutative space over* $\mathcal{V}$: $\mathcal{V}_{\text{nc}} = \coprod_{n=1}^{\infty} M_n(\mathcal{V})$
- *noncommutative sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that $X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$ for all $X \in \Omega_n$, $Y \in \Omega_m$, where $\Omega_n = \Omega \cap \mathcal{V}^{n \times n}$
- *upper admissible sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $Z \in \mathcal{V}^{n \times m}$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, with

$$\begin{bmatrix} X & \lambda Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Examples of upper-admissible sets:

- $\Omega = \mathrm{Nilp}\,\mathcal{V}$ = the set of nilpotent matrices over $\mathcal{V}$
- If $\mathcal{V}$ is a Banach space and Ω is open in the sense that $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open for all n, then Ω is upper admissible.
- $\mathcal{B}$ = unital C*-algebra, $\mathcal{A}$=$*$-algebra containing $\mathcal{B}$;
  $X \in \mathcal{A}$,
  $\rho_n(X : \mathcal{B}) = \{\beta \in M_n(\mathcal{B}) : 1_n \otimes X - \beta$ invertible$\}$
  $\rho(X;\mathcal{B}) = \coprod_{n=1}^{\infty} \rho_n(X : \mathcal{B})$ is upper admissible
- Noncommutative half-planes over $\mathcal{A}$:

$$\mathbb{H}^+(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a > 0\}$$
$$\mathbb{H}^-(\mathcal{A}_{\text{nc}}) = \{a \in \mathcal{A}_{\text{nc}} : \Im a < 0\}$$

$\Omega \subseteq \mathcal{V}_{\text{nc}}$ = non-commutative (upper admissible) set

Noncommutative function:
$f : \Omega \to \mathcal{W}_{\text{nc}}$ such that

- $f(\Omega_n) \subseteq M_n(\mathcal{W})$
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f respects similarities: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and $T \in \mathrm{GL}_n(\mathbb{C})$ such that $TXT^{-1} \in \Omega_n$.

Examples of nc-functions:

- *non-commutative polynomials*

  $\mathcal{V} = \mathcal{R}^m$, $\mathcal{W} = \mathcal{N}$, where $\mathcal{N}$ = module over a ring $\mathcal{R}$

  $f(X_1, \dots, X_m) = X_1X_3 - X_3X_1 + b_1X_2X_4B2X_5$

  N.b.: *A nc polynomial is determined uniquely by this type of nc function*

- $\mathbb{K}$ =field of characteristic zero

$$p_n = \sum_{\pi \in S_{n+1}} \text{sign}(\pi) X_1^{\pi(1)-1} X_2 \cdots X_1^{\pi(2)-1} X_2$$

$f : \Omega \in \mathbb{K}^2_{\text{nc}} \longrightarrow \mathbb{K}_{\text{nc}}$, given by

$$f(x_1, x_2) = \sum_{k=1}^{\infty} p_k(x_1, x_2).$$

For $(x_1, x_2) \in M_n(\mathbb{K}^2)$, the terms $p_k(x_1, x_2), k \geq n$ all vanish.
All n-dimensional components of f are polynomials, but f is **not** a nc polynomial.

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Examples:

- *the full $\mathcal{B}$-resolvent* of X
  on each $\rho(1_n \otimes X; M_n(\mathcal{B}))$ consider the analytic function $\beta \mapsto (1_n \otimes X - \beta)^{-1}$
- *the generalized Cauchy transform* of X: $\mathcal{G}_X$
  $\phi : \mathcal{A} \longrightarrow \mathcal{D}$ cp $\mathcal{B}$-bimodule map
  $\mathcal{G}_X = (G_X^{(n)})_n$, where

  $$G_X^{(n)} : \mathbb{H}^+(M_n(\mathcal{B})) \ni b \mapsto G_X^{(n)}(b) = \phi_n[(b - X \otimes 1_n)^{-1} \in \mathbb{H}^-(M_n(\mathcal{D}))$$
- the generalized moment series of $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$
  $\widetilde{\mu}\left((\mathbb{1} - \mathcal{X}b)^{-1}\right) = M_\mu = (M_{n,\mu})_n$, where $\widetilde{\mu} = (1_n \otimes \mu)_n$ id the fully matricial extension of μ.

  $$M_{n,\mu}(b) = \sum_{k=0}^{\infty} (1_n \otimes \mu)([\mathcal{X} \cdot b]^k),$$

M_X **indeed** encodes all the moments of X, not only the symmetric ones, since for

$$b = \begin{bmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in M_{n+1}(\mathcal{B})$$

we have that

$$(1_{N+1} \otimes \phi)\big([X \cdot b]^n\big) = \begin{bmatrix} 0 & \dots & 0 & \phi(Xb_1 \cdots Xb_n) \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

Difference-differential calculus

Nc functions admit a nice differential calculus. The difference-differential operators can be calculated directly by evaluation on block-triangular matrices.

$$f(\begin{bmatrix} X & Z \\ 0 & Y \end{bmatrix}) = \begin{bmatrix} f(X) & \Delta_R f(X,Y)(Z) \\ 0 & f(Y) \end{bmatrix}$$

The operator $Z \mapsto \Delta_R f(X,Y)(Z)$ is linear and

$$f(Y) = f(X) + \Delta_R(X,Y)(X-Y)$$

Particularly, if $\mathcal{B} \subset \mathcal{A}$, and $X \in \mathcal{A}$, a QD-bialgebra structure is induced by

$$\delta_X : \mathcal{B}\langle X\rangle \otimes \mathcal{B}\langle X\rangle \longrightarrow \mathcal{B}\langle X\rangle$$

where $\delta_{X|\mathcal{B}} = 0$ and $\delta_X(X) = 1 \otimes 1$.

For $f \in \mathcal{B}\langle X\rangle$, if $\delta_X(f) = \sum f_1 \otimes f_2$, with $f_1, f_2 \in \mathcal{B}\langle X\rangle$, then

$$\Delta_R f(X, X)(b) = \sum f_1 \cdot b \cdot f_2.$$

The relations for higher order derivatives is similar.

The Taylor-Taylor expansion:

If $f : \Omega \longrightarrow \mathcal{W}_{\text{nc}}$ is a non-commutative function, Ω=upper-admissible set, $X \in \Omega_n$. Then for each N and $X \in \Omega_n$ we have that

$$f(Y) = \sum_{k=0}^{N} \Delta_R^k f(\underbrace{X, \dots, X}_{k+1 \text{ times}})(\underbrace{X_Y, \dots, X - Y}_{k \text{ times}})$$

$$+\Delta_R^{N+1} f(\underbrace{X, \dots, X}_{N+1 \text{ times}}, Y)(\underbrace{X_Y, \dots, X - Y}_{N+1 \text{ times}})$$

XY

Moreover, if $0 \in \Omega$ then for $X \in \Omega$ we have

$$f(X) = \sum_{k=0}^{\infty} \widetilde{\Delta_R^k f}(\underbrace{0, \dots, 0}_{k+1})(\underbrace{X, \dots, X}_{k})$$

where $\widetilde{\Delta_R^k f}(\underbrace{0, \dots, 0}_{k+1})$ are the fully matricial extension of the multilinear maps $\Delta_R^k f(\underbrace{0, \dots, 0}_{k+1}) : \mathcal{V}^k \longrightarrow \mathcal{W}$.

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$$f\left(\begin{bmatrix} X & Z_1 & 0 & \cdots & 0 \\ 0 & X & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & X & Z_k \\ 0 & \cdots & \cdots & 0 & Y \end{bmatrix}\right)$$

$$= \begin{bmatrix} f(X) & \Delta_R f(X,X)(Z_1) & \cdots & \cdots & \Delta_R^k f(X,\ldots,X,Y)(Z_1,\ldots,Z_k) \\ 0 & f(X) & \ddots & & \Delta_R^{k-1} f(X,\ldots,X,Y)(Z_2,\ldots,Z_k) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f(X) & \Delta_R f(X,Y)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(Y) \end{bmatrix}.$$

Non-commutative R-transform

$\mathcal{B}$ = unital C*-algebra

$\mathcal{B} \subset \mathcal{A}$ inclusion of unital $*$-algebras

$\phi : \mathcal{A} \longrightarrow \mathcal{B}$ positive conditional expectation

$\mathcal{A} \in X^* = X \longleftrightarrow \mu_X \in \Sigma_{\mathcal{B}}$,

$$\mu_X(\mathcal{X}b_1\mathcal{X}b_2\cdots b_{n-1}\mathcal{X}) = \phi(Xb_1Xb_2\cdots b_{n-1}X)$$

$\boxplus : \mathcal{B}\langle\mathcal{X}\rangle \times \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}\langle\mathcal{X}\rangle$ given by
X, Y free over $\mathcal{B}$, then $\mu_{X+Y} = \mu_X \boxplus \mu_Y$

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The non-commutative R-transform of $\nu \in \Sigma_{\mathcal{B}}$:

$$M_\nu(b) - \mathbb{1} = R_\nu\left(bM_\nu(b)\right)$$

The equation is always meaningful for $b \in \mathrm{Nilp}(\mathcal{B})$;
if $\mathcal{A}$=C*-algebra, then also for b in a non-commutative ball around 0.

Properties:

- R_ν is a non-commutative function
- $R_{\mu \boxplus \nu} = R_\mu + R_\nu$
- The op-valued semicircular law is characterized by

$$R_{\mathfrak{s}}(b) = \eta(b)b$$

for some cp map $\eta : \mathcal{B} \longrightarrow \mathcal{B}$.

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Theorem (V.Vinnikov, M.P., '10)

$\mu \in \Sigma_{\mathcal{B}}$ is free infinitly divisible if and only if there exist some selfadjoint $\alpha \in \mathcal{B}$ and some completely positive ($\mathbb{C}$-)linear map $\nu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$ such that

$$R_\mu(b) = \left[\alpha \cdot \mathbb{1} + \widetilde{\nu}\left(b(\mathbb{1} - \mathcal{X}b)^{-1}\right)\right] b.$$

s cf i

The case of c. p. maps

C-freeness:

$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ subalgebras containing $\mathcal{B}$.
$\Phi : \mathcal{A} \longrightarrow \mathcal{D}$ a$\mathcal{B}$-bimodule map, $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ conditional expectation
$\mathcal{A}_1, \mathcal{A}_2$ are c-free if:

- $\mathcal{A}_1, \mathcal{A}_2$ are free w.r.t. ψ
- $\Phi(a_1 \cdots a_n) = \Phi(a_1) \cdots \Phi(a_n)$ whenever $\psi(a_j) = 0$ and $a_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(j) \neq \epsilon(j+1)$

b

logo

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b

- F. Boca (JFA, '91) - "amalgamated free product of cp maps"
- R. Speicher, M. Bozejko (Pac. J. Math, '96) - scalar case, "c-freeness", ${}^cR$-transform, limit laws
- K. Dykema, E. Blanchard (Pac. J. Math., '01) - reduced free products and embeddings of free products of von Neumann algebras
- M. P., J-C Wang ('08, to appear in Trans. AMS) - multiplicative properties, c-free S-transform

$$[{}^cT_X(m_X(z)-1)]\cdot M_X(z)=\frac{M_X(z)-1}{z}$$

- op-valued ${}^cR$-transform: W. Mlotkowsky ('03), M.P.('08)

bb

$$\Sigma_{\mathcal{B}:\mathcal{D}} = \{\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D} \ \ \mathcal{B}\text{-bimodule maps, c.p. }\}$$

$$\Sigma_{\mathcal{B}} = \{\nu : \mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow\longrightarrow \mathcal{B} \text{ positive conditional expectations }\}$$

$$\mathcal{A} \ni X = X^* \leftrightarrow (\mu_X, \nu_X) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$$

c-freeness induces

$$\boxed{\text{C}} : (\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \times (\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$$

$$((\mu_X, \nu_X), (\mu_Y, \nu_Y)) \mapsto (\mu_{X+Y}, \nu_{X+Y}), \ \ \text{where } X, Y = \text{c-free}$$

Op-valued Boolean independence:
$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are Boolean independent if

$\theta(a_1 a_2 \cdots a_n) = \theta(a_1) \cdots \theta(a_n)$ whenever $a_i \in \mathcal{A}_{\epsilon(i)}$ with $\epsilon(i) \neq \epsilon(i+1)$.

$$\uplus : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}:\mathcal{D}} \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}}$$

induced by $\mu_X \uplus \mu_Y = \mu_{X+Y}$, for X, Y=boolean independent
$\uplus$ is linearized by $\mu \longrightarrow B_\mu$,

$$B_\mu(b) \cdot M_\mu(b) = M_\mu(b) - \mathbb{1}$$

Op-valued Bernoulli law is given by $B_{\mathrm{Ber}} = \eta(b)b$;

- In general, $\mathfrak{s} \uplus \mathfrak{s} \neq \mathfrak{a}$
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Theorem (M.P.'10):

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- Any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$ is infinitely divisible with respect to boolean convolution.
- For any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$, there exists a selfadjoint $\alpha \in \mathcal{D}$ and a cp $\mathbb{C}$-linear map $\sigma : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$ such that

$$B_\mu(b) = \left[\alpha \cdot \mathbb{1} + \widetilde{\sigma}\left(b(\mathbb{1} - \mathcal{X}b)^{-1}\right)\right] \cdot b. \quad (1)$$

BP

Non-commutative ${}^cR$-transform for $(\mu,\nu)\in\Sigma_{\mathcal{B}:\mathcal{D}}\times\Sigma_{\mathcal{B}}$:

$$(M_\mu(b)-\mathbb{1})\cdot {}^cR_{\mu,\nu}(bM_\nu(b)) = (M_\mu(b)-\mathbb{1})\cdot M_\mu(b)$$

- ${}^cR$ is a non-commutative function
- if $X,Y\in\mathcal{A}$ are c-free w.r.t. (Φ,ψ), then ${}^cR_{X+Y}={}^cR_X+{}^cR_Y$

Theorem(M.P., V. Vinnikov '10)

A pair $(\mu,\nu)\in\Sigma_{\mathcal{B}:\mathcal{D}}\times\Sigma_{\mathcal{B}}$ is c-free infinitely divisible iff ν is free infinitely divisible and there exist some selfadjoint $\alpha\in\mathcal{B}$ and some completely positive ($\mathbb{C}$-)linear map $\nu:\mathcal{B}\langle\mathcal{X}\rangle\longrightarrow\mathcal{D}$ such that

$$ {}^cR_{\mu,\nu}(b)=\left[\alpha\cdot\mathbb{1}+\widetilde{\nu}\left(b(\mathbb{1}-\mathcal{X}\cdot b)^{-1}\right)\right]b.$$

r bp

Non-commutative ${}^cR$-transform for $(\mu, \nu) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$:

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r bp

The Non-commutative Boolean-to-Free Bercovici-Pata bijection:

$\mathcal{BP} : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}} \to$ $\boxed{\text{c}}$-infinitely divisible elements of $\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$$(\mu, \nu)$$

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$$\begin{array}{c}(\mu, \nu)\\ \Big\downarrow \\ (B_\mu, B_\nu)\end{array}$$

B-transf

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$$\begin{tikzcd}
(\mu, \nu) \arrow[d] & \\
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& (\alpha_1, \sigma_1, \alpha_2, \sigma_2)
\end{tikzcd}$$

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$$\begin{tikzcd}
(\mu, \nu) \arrow[d] & & \\
(B_\mu, B_\nu) \arrow[dr] & & ({}^{c}R_{\mu',\nu'}, R_{\nu'}) \\
& (\alpha_1, \sigma_1, \alpha_2, \sigma_2) \arrow[ur] &
\end{tikzcd}$$

R

The Non-commutative Boolean-to-Free Bercovici-Pata bijection:

$\mathcal{BP} : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}} \to$ $\boxed{\text{c}}$-infinitely divisible elements of $\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$$\begin{tikzcd}
(\mu, \nu) \arrow[d] & & (\mu', \nu') \\
(B_\mu, B_\nu) \arrow[dr] & & ({}^cR_{\mu',\nu'}, R_{\nu'}) \arrow[u] \\
& (\alpha_1, \sigma_1, \alpha_2, \sigma_2) \arrow[ur] &
\end{tikzcd}$$

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(B_\mu, B_\nu) \arrow[dr] & & ({}^cR_{\mu',\nu'}, R_{\nu'}) \arrow[u] \\
& (\alpha_1, \sigma_1, \alpha_2, \sigma_2) \arrow[ur] &
\end{tikzcd}$$

Idea: μ determines $\rho_\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$ via the moment-free cumulant recurrence relations. On $\mathcal{B}\langle\mathcal{X}\rangle_0 = \mathcal{B}\langle\mathcal{X}\rangle \setminus \mathcal{B}$ consider the positive pairing $\langle f(\mathcal{X}), g(\mathcal{X})\rangle = \rho_\mu(g(\mathcal{X})^* f(\mathcal{X}))$ and the self-adjoint operator $T : f(\mathcal{X}) \mapsto \mathcal{X} f(\mathcal{X})$.
Then consider the self-adjoint operator V on the full Fock $\mathcal{B}$-bimodule over $\mathcal{B}\langle\mathcal{X}\rangle_0$, which $\mathcal{B}$-valued distribution with respect to the ground $\mathcal{B}$-state coincides with μ, given by

$$V = a_X + a_X^* + \widetilde{T} + \mu(X)\mathrm{Id}.$$

The terms from the Taylor-Taylor development of $\widetilde{R}_\mu$ are the moments of $\widetilde{T}$ with respect to the mapping $\langle \cdot\mathcal{X}, \mathcal{X}\rangle$ and the conclusion follows using the additivity property of $\widetilde{R}$ and some cb-norm inequalities.

Theorem (Ş. Belinschi, M. P.)
Assume that $\{X_{jk}\}_{j\in\mathbb{N};1\leq k\leq k_j}$ is a triangular array of random variables in $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B}, \theta, \mathcal{D})$ of elements free (c-free, boolean independent) so that $\{X_{jk}: 1\leq k\leq k_j\}$ have the same distribution with respect to E_B (θ, E_B, θ) for each $j\in\mathbb{N}$ (i.e. rows are identically distributed).
Assume in addition that

$$\limsup_{j\to\infty}\|X_{j1}+\cdots+X_{jk_j}\|\leq M$$

for some $M\geq 0$. If $\lim_{j\to\infty} X_{j1}+X_{j2}+\cdots+X_{jk_j}$ exists in distribution as norm-limit of moments in $\Sigma_{\mathcal{B}}$ ($\Sigma_{\mathcal{B}:\mathcal{D}}\times\Sigma_{\mathcal{B}}$, $\Sigma_{\mathcal{B}:\mathcal{D}}$), then the limit distribution is free (c-free, boolean) infinitely divisible. bp

$\uplus : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}:\mathcal{D}} \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}}$ is linearized by $\mu \to B_\mu$,

$$M_\mu(b) - \mathbb{1} = B_\mu(b) \cdot M_\mu(b)$$

Theorem(M.P.'10):

1. Any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$ is infinitely divisible with respect to boolean convolution.
2. There exists a selfadjoint $\alpha \in \mathcal{D}$ and a cp $\mathbb{C}$-linear map $\sigma : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$ such that

$$B_\mu(b) = \left[\alpha \cdot \mathbb{1} + \widetilde{\sigma}\left(b(\mathbb{1} - \mathcal{X}b)^{-1}\right)\right] \cdot b. \qquad (2)$$

BP