Different Sized Haar-Unitaries arising from Random Matrix Models

Carlos Vargas Obieta[1]

[1]Fachrichtung Mathematik
Universität des Saarlandes

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Outline

A Random Matrix Model

Consider the following Random Matrix Model:

$$\Phi^{(N)} = \sum_{i=1}^{k} H_i U_i T_i U_i^* H_i^*,$$

where k is a fixed integer and:

- The U_i's are $N_i^{(N)} \times N_i^{(N)}$ Haar-distributed unitary matrices, drawn independently from $\mathscr{U}\left(N_i^{(N)}\right)$, $1 \leq i \leq k$.
- The H_i's are rectangular $N \times N_i^{(N)}$ deterministic matrices.
- The T_i's are non-negative diagonal $N_k^{(N)} \times N_k^{(N)}$ deterministic matrices.

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- $\left(N_1^{(N)}\right)_N, \dots \left(N_k^{(N)}\right)_N$ are sequences of numbers such that

$$\frac{N_i^{(N)}}{N} \to c_i \in (0,1),$$

$c_1 < \cdots < c_k$.

- The collection of deterministic matrices

$$(H_1, \dots H_k, T_1, \dots, T_k) \underset{N \to \infty}{\overset{*-dist}{\to}} (h_1, \dots, h_k, t_1, \dots, t_k),$$

with respect to the normalized expected trace $\mathbb{E} \circ \mathrm{tr}_N$.

PROBLEM: Describe the asymptotic distribution of $\Phi^{(N)}$.

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SOLUTION: [Coulliet, Debbah, Hoydis 2010]:
The (implicit) system of equations

$$\bar{e}_i(z) = \operatorname{tr}_N T_i \left(e_i(z) T_i + (1 - e_i(z)\bar{e}_i(z)) I_{N_k}\right)$$

$$e_i(z) = \operatorname{tr}_N H_i H_i^* \left(\sum_{j=1}^{k} \bar{e}_j(z) H_j H_j^* - z I_N\right)^{-1}$$

has a unique solution $(\bar{e}_1, \ldots, \bar{e}_k)$ in the class of k-tuples of Stieltjes transforms of positive measures.

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has a unique solution $(\bar{e}_1, \ldots, \bar{e}_k)$ in the class of k-tuples of Stieltjes transforms of positive measures.

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Then the function

$$\bar{m}(z) := \mathrm{tr}_N \left(\sum \bar{e}_i(z) H_i H_i^* - zI_N\right)^{-1}$$

is a **deterministic equivalent** equivalent for the Stieltjes Transform $m(z)$ of $\Phi^{(N)}$.

This means that

$$|\bar{m}(z) - m(z)| \quad \underset{N\to\infty}{\overset{a.s.}{\to}} \quad 0.$$

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Question

- Can we tackle this problem using free probability?

"Answer"

MAYBE:

- The case when all the matrices have the same size can be treated.
- There is a "free-probabilistic" object that produces the equations of the deterministic equivalent
- Different sized unitaries are not free ($k \geq 3$), however
- they have nice operator valued distributions w.r.t a conditional expectation.

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Haar-Unitaries and R-diagonal Elements

Let $(\mathscr{A}, \tau)$ be a tracial W^*-probability space.

Definition

An element $u \in \mathscr{A}$ is called Haar-Unitary, if $uu^* = 1 = u^*u$ and for every $k \neq 0$

$$\tau\left(u^k\right) = 0.$$

Definition (Nica, Speicher 97)

An element $r \in \mathscr{A}$ is called R-Diagonal, if every non-alternating $*$-cumulant on r vanishes. That is, for $(\varepsilon_1, \ldots, \varepsilon_n) \in \{1, *\}^n$,

$$\kappa_n\left(r^{\varepsilon_1}, r^{\varepsilon_2}, \ldots, r^{\varepsilon_n}\right) = 0,$$

unless $n = 2k$ and $\varepsilon_1 \neq \varepsilon_2 \neq \cdots \neq \varepsilon_n$.

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Basic Definitions in Op. Val. FP

Haar-Unitaries of Different size

Let $(\mathscr{A}, \tau)$ be a tracial W^*-probability space. Let $a_1, \ldots, a_k, a_{k+1}, p_1, \ldots, p_{k+1} \in \mathscr{A}$ be such that

- $p_1, \ldots, p_{k+1}$ are pairwise orthogonal projections, s.t. $\tau(p_i) > 0$, $p_1 + \cdots + p_{k+1} = 1$. We set $q_i = p_1 + \cdots + p_i$.
- $a_i \perp (1 - q_i)$, a_i is free from $a_1, \ldots, a_{i-1}, p_1, \ldots, p_i$ in the compressed space $\left(q_i \mathscr{A}_i q_i, \tau(q_i)^{-1} \tau|_{q_i \mathscr{A}_i q_i}\right) =: (\mathscr{A}_i, \tau_i)$

Theorem

$a_1, \ldots, a_{k+1}$ are free with amalgamation over $\langle p_1, \ldots, p_{k+1} \rangle =: \mathscr{B}$.

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Haar-Unitaries of Different size

Our elements $\tilde{t}_i := u_i t_i u_i^*, \tilde{h}_i := U h_i v_i, i = 1, \ldots, k \in \mathscr{A}$ fit into the framework of the previous theorem and we obtain:

1). $\tilde{t}_1, \ldots, \tilde{t}_k, \tilde{h}_1^* \tilde{h}_1, \ldots, \tilde{h}_k^* \tilde{h}_k, \left\langle \tilde{h}_1 \tilde{h}_1^*, , \ldots, \tilde{h}_k \tilde{h}_k^* \right\rangle$ are free with amalgamation over $\mathscr{B}$.

2). The $\tilde{h}_i$'s are $\mathscr{B}$-valued R-diagonal elements.

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One Possibility

We would wish that actually $\left\langle \tilde{h_1}, \ldots, \tilde{h_k} \right\rangle$, $\langle \tilde{t}_1, \ldots, \tilde{t}_k \rangle$ are free with amalgamation over $\mathscr{B}$

If so, by setting $\mathscr{D} := \left\langle \tilde{h_1}, \ldots, \tilde{h_k}, p_1, \ldots, p_{k+1} \right\rangle$ a straight-forward use of the Characterization of Freeness in different operator-valued levels by Nica, Shlyakhtenko and Speicher gives that

$$\tilde{h_1} \tilde{t}_1 \tilde{h_1}, \ldots, \tilde{h_k} \tilde{t}_k \tilde{h_k}$$

are $\mathscr{D}$-free, moreover

$$\kappa_n^{\mathscr{D}} \left(\tilde{h_i} \tilde{t}_i \tilde{h_i}^* \right) = \left(\frac{\tau\left(h_i h_i^*\right)}{c_i} \right)^{n-1} \kappa_n^{\mathbb{C}} \left(t_i, \ldots, t_i\right) h_i h_i^* .$$

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Conclusion

After some algebraic manipulationd and the use of the Operator Valued Subordination Formula, we would obtain

$$G_{\Phi}^{\mathscr{D}}(z) = \left(\sum_{i=1}^{k} \mathscr{R}_{t_i}^{\mathbb{C}} \left(\frac{\tau(h_i h_i^*)\, G_{\Phi}^{\mathscr{D}}(z)}{c_i} \right) h_i h_i^* \right)^{-1},$$

Finally, by applying τ we obtain an equation for the Scalar Cauchy transform. (the solution $(e_1(z), \ldots, e_k(z))$ from Debbah et al. would just be

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Block Matrices

Reduction of the problem to the block matrix case

Nested Free Compressions

We recall the calculation of compressed cumulants

Theorem (Nica, Speicher)

Let $a, p \in \mathscr{A}$ be free elements, such that p is a projection with $\tau(p) > 0$. Let κ^p be the free cumulants in the compressed space $\left(p\mathscr{A}p, \tau(p)^{-1}\tau|_{p\mathscr{A}p}\right)$ Then, for $\pi \in NC(n)$

$$\kappa_\pi^{(p)}(pap, \ldots, pap) = \kappa_\pi(a, \ldots, a)\,\tau(p)^{n-|\pi|}.$$

As a nice corollary one can show that if $a, b, p \in \mathscr{A}$ are free, then pap, pbp are free in $\left(p\mathscr{A}p, \tau(p)\tau|_{p\mathscr{A}p}\right)$.

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We should analyze the same for other matrix units, $e_{1i}ae_{j1}, e_{1k}be_{l1}$. (Note that $e_{j1}e_{1k} = e_{jk}$).

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As a consequence of the previous observation, and a bit of work can show that the upper left blocks of our matrices are free.

By the unitary invariance of our matrices, this suggest that actually the algebras generated by the small blocks should be free.

Such a result would lead to the same conclusion for the asymptotics of the Cauchy Transform of our matrix Φ.

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