Technische Universität Graz

Institut für Mathematische Strukturtheorie

Asymptotic Behavior of Random Walks on Free Products of Groups

Vienna – March, 28th 2011

Elisabetta Candellero (joint work with **Lorenz A. Gilch**)

Overview on Fundamental Concepts

Environment

Introduction of the structure we work on: the Free Product Γ.

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We consider a Nearest Neighbor Random Walk on Γ.

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Find the asymptotic behavior of the Return Probabilities of the RW.

Method

We find the Expansion of the Green Function to use Darboux's method.

Example of Free Product of two Finite Groups

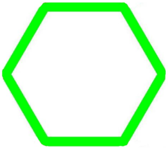

Figure: Example: $\mathbb{Z}_6 * \mathbb{Z}_3$

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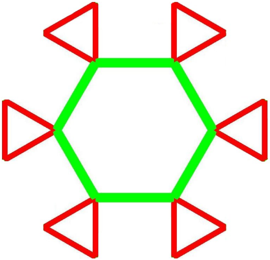

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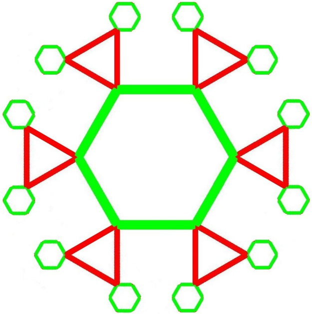

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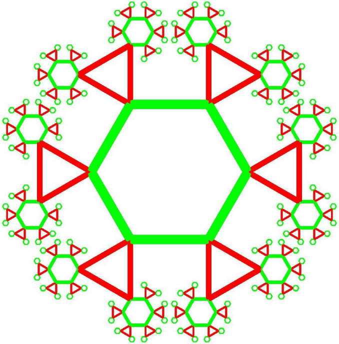

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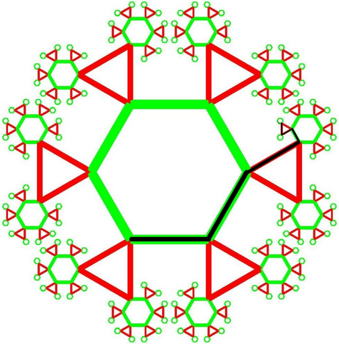
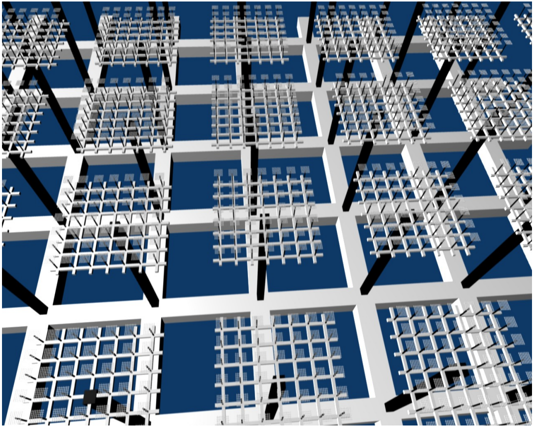


Figure: Example: $x_1x_2x_3x_4x_5$ is an element of $\mathbb{Z}_6 * \mathbb{Z}_3$

Example of Free Product of a Finite and an Infinite Group



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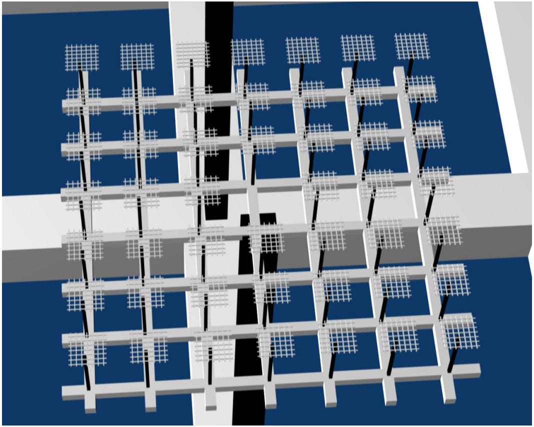

Figure: Example: Video where we see the construction of $\mathbb{Z}^2 * \mathbb{Z}_2$

Formal Overview: define a RW

$\Gamma :=$ group with identity e. $A :=$ set of generators of Γ; $|A| < \infty$.
$\mu :=$ probability measure on Γ: defines a RW with transition probabilities

$$\forall x, y \in \Gamma \qquad p(x,y) = \mu(x^{-1}y); \qquad (p(x,y) > 0 \text{ iff } x^{-1}y \in A).$$

$\mu^{(n)}(x^{-1}y) := p^{(n)}(x,y) =$ probability to go from x to y in n steps.
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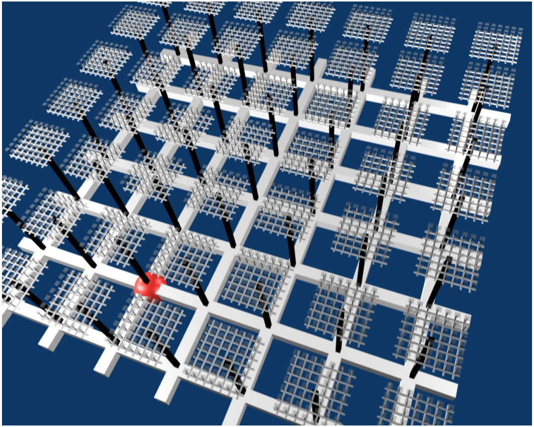
Asymptotic Behavior of the Return Probabilities $\mu^{(n)}(e)$

In a great variety of cases:

$$\mu^{(n)}(e) \sim C \cdot \mathbf{r}^{-n} n^{-\lambda},$$

where $1/\mathbf{r} \leq 1$ is the "spectral radius" and $\lambda > 0$ a parameter depending on the structure of Γ and on the RW.

Idea of a Random Walk on $\mathbb{Z}^2 * \mathbb{Z}_2$



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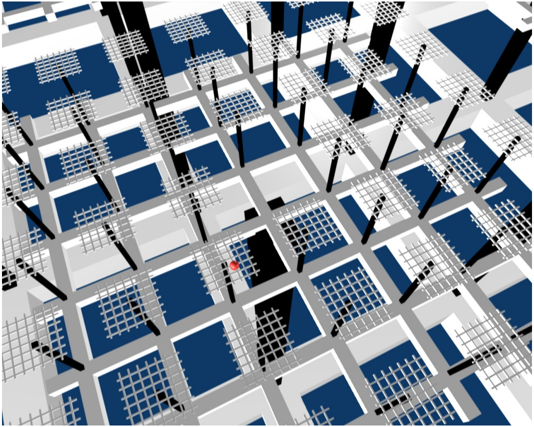

Figure: Example: Video where we see an example of a RW on $\mathbb{Z}^2 * \mathbb{Z}_2$

Background and Motivation

Gerl's conjecture

Gerl [Ger81]: the n-step return probabilities of two symmetric measures on a group have the same $n^{-\lambda}$. I.e. λ is a group invariant.

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Cartwright [Car89]: found a *counterexample*. There exist at least two probability measures (with same support) on $\mathbb{Z}^d * \mathbb{Z}^d$ with $d \geq 5$ with different asymptotic behaviors: $n^{-3/2}$ and $n^{-d/2}$.

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Questions

- What are the motivations of Cartwright's examples? What are the possible asymptotic behaviors on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ $(d_1 \neq d_2)$?
- What happens on $\Gamma_1 * \Gamma_2$ (Γ_1 and Γ_2 finitely generated groups)?

Definitions

Starting Objects

- $\Gamma_1, \ldots, \Gamma_m$: finitely generated groups with identities $\{e_i\}_{i=1}^m$;
- $\mu_1, \ldots, \mu_m$: probability measures s.t. $\langle \mathrm{supp}(\mu_i) \rangle = \Gamma_i$.

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Structure and Properties

- Free Product $\Gamma := \Gamma_1 * \ldots * \Gamma_m$: the set of all finite words of the form $x_1 x_2 \ldots x_n$, where $x_1, \ldots, x_n$ are elements of $\bigcup_i \Gamma_i \setminus \{e_i\}$ and x_j, x_{j+1} do not belong to the same group.

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- Define the probability measure on Γ

$$\mu := \alpha_1 \mu_1 + \alpha_2 \mu_2 + \ldots + \alpha_m \mu_m,$$

s.t. $\sum_{i=1}^m \alpha_i = 1$ and $\alpha_i > 0$ for every index $i \in \{1, \ldots m\}$.
We consider a RW on Γ governed by μ.

Green Functions

Green Functions

- $G_i(z) := \sum_{n=0}^{\infty} \mu_i^{(n)}(e_i) z^n$ on the free factors Γ_i for $i = 1, \ldots, m$;
- analogously on Γ we have $G(z) := \sum_{n=0}^{\infty} \mu^{(n)}(e) z^n$.

The radii of convergence will be denoted by $\mathbf{r}_i$ and $\mathbf{r}$ respectively.
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Idea

We find the singular expansion of $G(z)$ near $z = \mathbf{r}$ and then apply the Darboux's Method.

Remark: It is possible to use another method known as “Singularity Analysis” (see [FO90] and [FS09]), but there is no advantage here.

Full Classification of RWs on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$, $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$

Let us compute the singular term of the Green function for each factor $\mathbb{Z}^d, d \geq 1$:

$$S_d(z) \sim \begin{cases} (\mathbf{r}_d - z)^{(d-2)/2}, & \text{if } d \text{ is odd,} \\ (\mathbf{r}_d - z)^{(d-2)/2} \log(\mathbf{r}_d - z), & \text{if } d \text{ is even,} \end{cases}$$

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There are three possible cases:

1st possibility: first singular term of $G(z)$ (defined on $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$) is proportional to

$$\sqrt{\mathbf{r} - z}. \tag{1st}$$

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In this case:

$$\mu^{(2n)}(e) \sim C_2 \cdot \mathbf{r}^{-2n} \cdot n^{-d_2/2}.$$

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The Functional Equation

The trick to understand what happens, is to consider a functional equation (concerning $G(z)$), seen as a function of the parameter α_1.

Consider

$$\Psi(zG(z)) = \frac{1}{1 - U(z) + zU'(z)} \ ,$$

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where $U(z) := 1 - 1/G(z)$. (The dependence on α_1 is hidden!)
There are 3 possibilities: at a (known, it comes from the computations) critical value $t = \bar{\theta}$, the function $\Psi(t)$ can be

1. $\Psi(\bar{\theta}) < 0$; 2. $\Psi(\bar{\theta}) > 0$; 3. $\Psi(\bar{\theta}) = 0$.

The Functional Equation

Case $\Psi(\bar{\theta}) < 0$

This is already known (e.g. [CS86], [Car88], [Woe00]): $\mu^{(n)}(e) \sim C \cdot \mathbf{r}^{-n} n^{-3/2}$.

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Case $\Psi(\bar{\theta}) > 0$

Theorem: If $\Psi(\bar{\theta}) > 0$, there are two possible behaviors:

$$\mu^{(n)}(e) \sim \begin{cases} C_1 \cdot \mathbf{r}^{-n} n^{-d_1/2} & \text{if } \alpha_1 \geq \alpha_c \\ C_2 \cdot \mathbf{r}^{-n} n^{-d_2/2} & \text{if } \alpha_1 < \alpha_c. \end{cases}$$

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Case $\Psi(\bar{\theta}) = 0$

Theorem: $\Psi(\bar{\theta}) = 0$ yields $\mu^{(n)}(e) \sim C \cdot \mathbf{r}^{-n} n^{-3/2}$.

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The function $\Upsilon(\alpha_1) := \Psi(\bar{\theta})$, seen as a function of α_1 behaves approximately like a truncated parabola, in particular it can have $0, 1$ or 2 zeros, according to its characteristics.

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- Otherwise, the asymptotic behavior obeys the $n^{-3/2}$-law.

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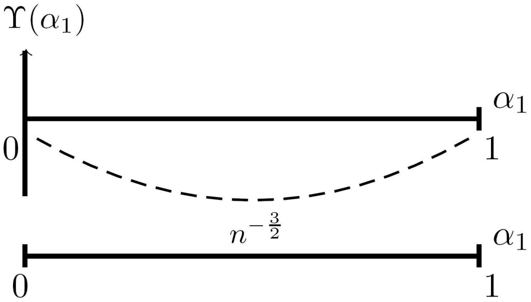

Figure: 1st Case: no phase transition.

The Functional Equation

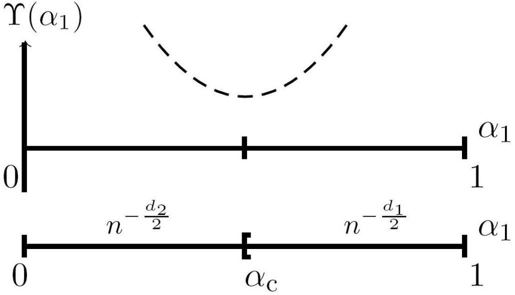

Figure: 2nd Case, e.g if μ_1 and μ_2 are Simple RWs.

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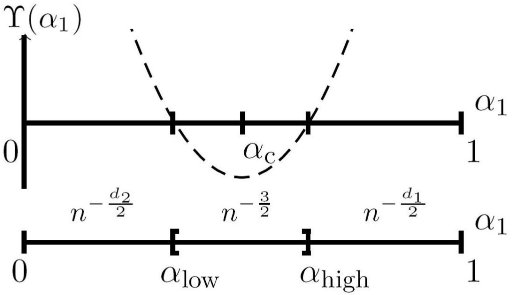
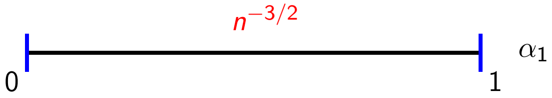


Figure: General Case (2 phase transitions).

Complete Picture from the Point of View of α_1

Suitable μ_1 and μ_2



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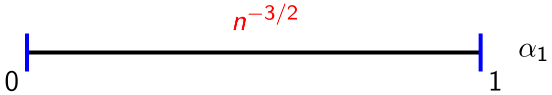
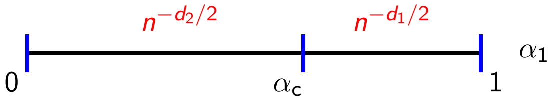


μ_1 and μ_2 Simple RWs

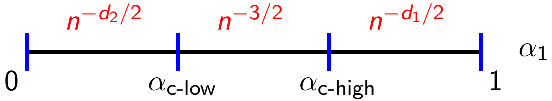


Here there is a value α_c which determines a *phase transition*.

Complete Picture from the Point of View of α_1

For some μ_1 and μ_2 it is possible to obtain all three behaviors, just depending on the value of the parameter α_1.

Keep μ_1, μ_2 fixed, α_1 varies: all possible behaviors



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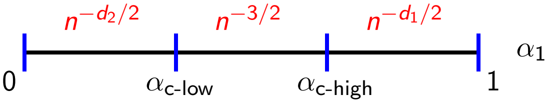

Possible Combinations

It is possible to have one or two sub-intervals collapsing, depending on the properties of the Functional Equation concerning $G(z)$.

Asymptotics

What is the meaning of our result?

If $\Psi(\bar{\theta}) > 0$, the RW on Γ inherits its (non-exponential) behavior either from the RW defined on $\mathbb{Z}^{d_1}$ or from the RW defined on $\mathbb{Z}^{d_2}$.

Otherwise we have the $n^{-3/2}$-behavior.

More general Groups

On $\Gamma := \Gamma_1 * \Gamma_2$

Assume the $G_i(z)$ have *algebraic* or *logarithmic* singular expansion. Then *up to* 3 different asymptotic behaviors are possible for $\mu^{(n)}(e)$:

$$C \cdot \mathbf{r}^{-n} n^{-3/2}, \qquad C_1 \cdot \mathbf{r}^{-n} n^{-\lambda_1} \log^{\kappa_1} n, \qquad C_2 \cdot \mathbf{r}^{-n} n^{-\lambda_2} \log^{\kappa_2} n.$$

($\lambda_1, \lambda_2 > 0$, $\kappa_1, \kappa_2 \geq 0$ are parameters related to the singular expansions of $G_1(z)$ and $G_2(z)$ respectively).

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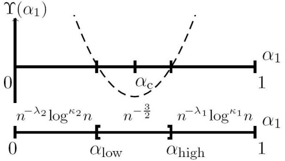

Figure: General Case

Résumé

Theorem: General Result (C. and Gilch)

Define $\Gamma := \Gamma_1 * \ldots * \Gamma_m$, by induction we find:

If the Green Function on the Free Factors have **algebraic** *or* **logarithmic** *singularity, then the asymptotic behavior of $\mu^{(n)}(e)$ on Γ obeys one of the following laws:*

$$\mu^{(n)}(e) \sim \begin{cases} C_i \mathbf{r}^{-n} n^{-\lambda_i} \log^{\kappa_i} n & \text{for one } i \in \{1, \ldots, m\} \\ C_0 \mathbf{r}^{-n} n^{-3/2} & \end{cases}$$

According to the positivity of the functional equation we can observe that the RW on Γ:

- either inherits its (non-exponential) behavior from the RW defined on one of the factors,
- or obeys the $n^{-3/2}$-law.

Idea of the used Method: Case $\Psi(\bar{\theta}) > 0$

- Definition of $\Psi(t) \Rightarrow$ if $\Psi(\bar{\theta}) > 0$ then $G'(\mathbf{r}) < \infty \Rightarrow$ no square-root!

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- We find the singular expansion of $G(z)$ near $z = \mathbf{r}$: it has the same form as $\xi_j(z)$.
- Once the expansion of $G(z)$ is known, apply Darboux's method:

$$\mu^{(n)}(e) \sim \mathbf{r}^{-n} n^{-\lambda_j} \log^{\kappa_j}(n).$$

Idea of the used Method: Case $\Psi(\bar{\theta}) = 0$

It is possible to prove that $\Psi(\bar{\theta}) = 0$ leads to some properties of the function $G(z)$ such that

$$\lim_{z \to \mathbf{r}} \frac{(G(z) - G(\mathbf{r}))^2}{\mathbf{r} - z} < \infty, \quad \text{or, equivalently,} \quad G(z) - G(\mathbf{r}) \sim \sqrt{\mathbf{r} - z}.$$

Expanding a bit further, we can apply Darboux's method, obtainig

$$\mu^{(n)}(e) \sim \mathbf{r}^{-n} n^{-3/2}.$$

Skip Darboux

Darboux's Method

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$$G(z) = S(z) + R(z).$$

Known: asymptotic Taylor expansion of $S(z) = \sum_{n=0}^{\infty} a_n z^n$ near $z = 0$ (when $S(z)$ has algebraic or logarithmic terms: $a_n \sim \mathbf{r}^{-n} n^{-k}$, for some suitable $k > 0$).

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If $(*)$ is not satisfied, we **have to** expand $G(z)$ further ($S(z)$ will contain "enough" terms), until it holds.

Remarks

- The method of Singularity Analysis, developed by Flajolet and Sedgewick [FS09], seems to be of easier application, because we just need to know the *first* singular term instead of a more precise expansion. BUT the problems we had to face in order to find a further expansion, arised as well while trying to verify that the function $G(z)$ had an analytic continuation outside its circle of convergence.

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- Given any $\Gamma = \Gamma_1 * \ldots * \Gamma_m$ (with at least one element of order ≥ 3), it is always possible (see e.g. [Woe00]) to find a set of measures $\mu_1, \ldots, \mu_m$ s.t. $\mu^{(n)}(e) \sim C \cdot \mathbf{r}^{-n} n^{-3/2}$. This does not apply to the other behaviors.

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- On $\mathbb{Z}^{d_1} * \mathbb{Z}^{d_2}$ we have either $n^{-3/2}$ or $n^{-d_i/2}$ (with $d_i \geq 5$), it follows that we can *never* have n^{-2}.

Next Work

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Thank you for your Attention!

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