Convolution semigroups for operator-valued distributions

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Scalar-valued context.

Let $X = X^* \in (\mathcal{A}, \tau)$ be a random variable in a non-commutative probability space. We denote by μ_X its distribution with respect to τ. Some analytic tools:

- The Cauchy transform

$$G_{\mu_X}(z) = G_X(z) = \tau\left[(z - X)^{-1}\right], \quad \Im z \neq 0;$$

- The R-transform:

$$R_X(z) = G_X^{-1}(z) - \frac{1}{z}, \quad |z| \text{ small}, \left|\frac{\Re z}{\Im z}\right| \text{ bounded}.$$

Relevance: if $X_1, X_2, \ldots, X_t$ are free, then (Voiculescu '86)

$$R_{X_1}(z) + R_{X_2}(z) + \cdots + R_{X_t}(z) = R_{X_1+X_2+\cdots+X_t}(z).$$

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Notation/theorem (Voiculescu): $\mu_{X_1+X_2} = \mu_{X_1} \boxplus \mu_{X_2}$. If $\mu_{X_1} = \cdots = \mu_{X_t}$, we shall write $\mu_{X_1} \boxplus \cdots \boxplus \mu_{X_t} = \mu_{X_1}^{\boxplus t}$. Note:

$$R_{\mu_{X_1}^{\boxplus t}}(z) = tR_{\mu_{X_1}}(z). \tag{1}$$

What if $t \notin \mathbb{N}$? Unlike in classical probability, for any $t \in [1, +\infty)$ and probability measure μ on $\mathbb{R}$, there exists a probability $\mu^{\boxplus t}$ on $\mathbb{R}$ so that (1) holds; $\{\mu^{\boxplus t} : t \geq 1\}$ forms a partial semigroup.

- Proved first by Bercovici and Voiculescu for large $t > 1$ with R-transform methods (1995);
- Then by Nica and Speicher for any $t \geq 1$ through a specific operatorial construction (1998): if $p_t = p_t^* = p_t^2$ is free from X and $\tau(p_t) = 1/t$, then $\mu_{p_t X p_t} = (1 - \frac{1}{t})\delta_0 + \frac{1}{t} D_{\frac{1}{t}}(\mu_X^{\boxplus t})$.

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Like in classical probability, there exists a special class of distributions μ for which $\mu^{\boxplus t}$ makes sense for any $t \geq 0$, namely the *freely infinitely divisible* ones.

Theorem

- *(Bercovici-Voiculescu) A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if there are a $a \in \mathbb{R}$ and ρ a positive measure so that*

$$R_\mu(1/z) = a - \int_{\mathbb{R}} \frac{1+tz}{z-t}\, d\rho(t), \quad z \notin \mathbb{R}.$$

- *(Bercovici-Pata) A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if it is the distributional limit of an infinitesimal array of free random variables.*

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Operator-valued distributions and fully matricial maps

Definition

(Voiculescu) Operator-valued non-commutative probability space: $(\mathcal{A}, \mathbb{E}_B, B)$, where $\mathcal{A}$ is a unital C^*-algebra, B is a C^*-subalgebra of $\mathcal{A}$ containing the unit of $\mathcal{A}$, and $\mathbb{E}_B \colon \mathcal{A} \to B$ is a unit-preserving *conditional expectation*.

Distribution of $X \in \mathcal{A}$ with respect to $\mathbb{E}_B$:

$\mu_X = \{m_n^X \colon B^{n-1} \to B : m_n^X(b_1, \ldots, b_{n-1}) = \mathbb{E}_B(Xb_1 \cdots Xb_{n-1}X)\}.$

Operator valued Cauchy-Stieltjes transform of an $X = X^* \in \mathcal{A}$:

$$G_X(b) = \mathbb{E}_B\left[(b - X)^{-1}\right], \quad \Im b \geq \varepsilon 1$$

for some $\varepsilon > 0$ (call this $\Im b > 0$). Note:

$$\Im G_X(b) < 0, \quad \lim_{\|b^{-1}\| \to 0} bG_X(b) = \lim_{\|b^{-1}\| \to 0} G_X(b)b = 1.$$

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$G_X(b) = \mathbb{E}_B\left[(b - X)^{-1}\right] = \sum \mathbb{E}_B(b^{-1}(Xb^{-1})^n), \ \|b^{-1}\| < \frac{1}{\|X\|}$, so $G_X(b)$ allows us to recover the *symmetric moments* of X: $m_n(b^{-1}, \ldots, b^{-1}) = \mathbb{E}_B(Xb^{-1}X \cdots b^{-1}X)$.

Complete positivity, fully matricial maps (Voiculescu): $\mathbb{E}_B \otimes \mathrm{Id}_{\mathcal{M}_n(\mathbb{C})} : \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \to B \otimes \mathcal{M}_n(\mathbb{C})$ remains positive. (Call $\mathbb{E}_B \otimes \mathrm{Id}_{\mathcal{M}_n(\mathbb{C})} = \mathbb{E}_{\mathcal{M}_n(B)}$.) Can define

$$G_{X\otimes 1_n}(b) = \mathbb{E}_{\mathcal{M}_n(B)}\left[(b - X \otimes 1_n)^{-1}\right], \quad b \in \mathcal{M}_n(B), \Im b > 0,$$

$$G_{X\otimes 1_n}(b) = \sum_{n=0}^{\infty} \mathbb{E}_{\mathcal{M}_n(B)}(b^{-1}((X \otimes 1_n)b^{-1})^n)$$

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Meaning: if $b_1, \ldots, b_n$ are given, and

$$b = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \text{then}$$

$$\mathbb{E}_{\mathcal{M}_n(B)}(b((X \otimes 1_{n+1})b)^{n-1}) = \begin{pmatrix} 0 & 0 & \cdots & \mathbb{E}_B(b_1 X b_2 \cdots X b_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

and $\mathbb{E}_B \otimes \mathrm{Id}_{\mathcal{M}_{n+1}(\mathbb{C})}(b((X \otimes 1_{n+1})b)^j) = 0$ for $j \geq n$. Set $b_1 = b_n = 1$ to get $m_{n-1}^X(b_2, \ldots, b_{n-1})$

Operator-valued distributions and fully matricial maps

Free convolution: (Voiculescu '95) If $(\mathcal{A}, \mathbb{E}_B, B)$ is an op-valued noncommutative probability space and $X = X^*$, $Y = Y^* \in \mathcal{A}$ is free over B, then $\mu_{X+Y} = \mu_X \boxplus \mu_Y$ is determined by μ_X and μ_Y and is called the *free additive convolution* of μ_X and μ_Y.

Analytic tools for the study of $\mu_X \boxplus \mu_Y$:

1. Op-valued R-transform: $R_{X\otimes 1_n}(b) = G^{-1}_{X\otimes 1_n}(b) - b^{-1}$ satisfies
$$R_{\mu_X \boxplus \mu_Y}(b) = R_{\mu_X}(b) + R_{\mu_Y}(b), \quad \|b\| \text{ small.}$$

2. *Analytic subordination* property: there exists a unique fully matricial self-map $\omega_1^{(n)}$ of the upper half-plane $\mathcal{M}_n(B)^+$ so that
$$\mathbb{E}_{\mathcal{M}_n(B[X])}\left((b - (X+Y)\otimes 1_n)^{-1}\right) = (\omega_1^{(n)}(b) - X \otimes 1_n)^{-1}.$$

By applying $\mathbb{E}_{\mathcal{M}_n(B)}$, we get $G_{X\otimes 1_n} \circ \omega_1^{(n)} = G_{(X+Y)\otimes 1_n}$.

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Operator-valued free infinite divisibility

Definition

(Speicher) The B-valued distribution μ_X is $\boxplus$-infinitely divisible if for any $n \in \mathbb{N}$ there exist B-free identically distributed selfadjoint B-valued random variables $X_1, \dots, X_n$ so that

$$\mu_X = \mu_{X_1 + \cdots + X_n} = \mu_{X_1}^{\boxplus n}.$$

Few examples: an op-valued CLT (op-valued semicircle law) of Voiculescu ('95) and an op-valued free Poisson (Speicher).

Theorem

(M. Popa, V. Vinnikov) The B-valued distribution μ is $\boxplus$-infinitely divisible if and only if there are $a = a^* \in B$ and $\rho \colon B[X] \to B$ completely positive map so that

$$R_\mu(b^{-1}) = a + \sigma\left[(b - X)^{-1}\right], \Im b > 0.$$

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A limit theorem

Theorem

Assume that $\{X_{jk}: j\in\mathbb{N}, 1\leq k\leq k_j\}$ *is a triangular array of selfadjoint random variables in* $(\mathcal{A},\mathbb{E}_B,B)$ *with* identically distributed rows. *Assume in addition that* $\limsup_{j\to\infty}\|X_{j1}+\cdots+X_{jk_j}\|\leq M$ *for some fixed* $M\geq 0$. *If* $\lim_{j\to\infty}X_{j1}+\cdots+X_{jk_j}$ *exists in distribution as norm-limit of moments, then the limit distribution is freely infinitely divisible.*

Idea of proof: note that, as in the scalar case, $G_{X_{j1}+\cdots+X_{jk_j}}(b)=G_{\mu_{X_{j1}}^{\boxplus k_j}}(b)=G_{\mu_{X_{j1}}}(\omega_j(b))$ (Voiculescu's subordination) and $b=k_j\omega_j(b)+(k_j-1)G_{\mu_{X_{j1}}}(\omega_j(b))^{-1}$ (R-transform). So (i) $\omega_j(b)=G_{\nu_j}(b)^{-1}$, where $\nu_j=(\mu_{X_{j1}}^{\boxplus k_j})^{\uplus 1-\frac{1}{k_j}}$, and (ii) the R-transform of ν_j is $R_{\nu_j}(b^{-1})=(k_j-1)(b-G_{\mu_{X_{j1}}}(b)^{-1})$.

A limit theorem

Theorem

Assume that $\{X_{jk}: j \in \mathbb{N}, 1 \leq k \leq k_j\}$ *is a triangular array of selfadjoint random variables in* $(\mathcal{A}, \mathbb{E}_B, B)$ *with* identically distributed rows. *Assume in addition that* $\limsup_{j\to\infty} \|X_{j1} + \cdots + X_{jk_j}\| \leq M$ *for some fixed* $M \geq 0$*. If* $\lim_{j\to\infty} X_{j1} + \cdots + X_{jk_j}$ *exists in distribution as norm-limit of moments, then the limit distribution is freely infinitely divisible.*

Idea of proof: note that, as in the scalar case, $G_{X_{j1}+\cdots+X_{jk_j}}(b) = G_{\mu_{X_{j1}}^{\boxplus k_j}}(b) = G_{\mu_{X_{j1}}}(\omega_j(b))$ (Voiculescu's subordination) and $b = k_j\omega_j(b) + (k_j - 1)G_{\mu_{X_{j1}}}(\omega_j(b))^{-1}$ (R-transform). So (i) $\omega_j(b) = G_{\nu_j}(b)^{-1}$, where $\nu_j = (\mu_{X_{j1}}^{\boxplus k_j})^{\uplus 1-\frac{1}{k_j}}$, and (ii) the R-transform of ν_j is $R_{\nu_j}(b^{-1}) = (k_j - 1)(b - G_{\mu_{X_{j1}}}(b)^{-1})$.

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Aside: One can show, using Dykema's version of the S-transform, that under the hypotheses that $\mathbb{E}_B(p)$ is invertible and belongs to the centre of B, that

$$\mathbb{E}_B(p)^{-1}\mathbf{R}_{D_{\mathbb{E}_B(p)}(\mu_X)}(\mathbb{E}_B(p)^{-1}b) =$$
$$\mathbb{E}_B(p)^{-1}\Psi_{Xp}\left(\mathbb{E}_B(p)^{-1}b[1+\mathbb{E}_B(p)^{-1}\mathbf{R}_{D_{\mathbb{E}_B(p)}(\mu_X)}(\mathbb{E}_B(p)^{-1}b)]^{-1}\right),$$

which is essentially the proof of Nica and Speicher from the scalar-valued context.

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Free convolution semigroups indexed by cp maps

This (among others) suggests “generalizing” $t > 1$ to a cp map:

Theorem

For any cp map $\alpha\colon B \to B$ so that $\alpha - 1$ is still cp and invertible, and for any B-valued selfadjoint random variable X, there exists a B-valued selfadjoint random variable X_α so that

$$\alpha(R_{\mu_X}(b)) = R_{\mu_{X_\alpha}}(b).$$

Denote μ_{X_α} by $\mu_X^{\boxplus\alpha}$.

One (rough) idea of proof: (i) show that $\mu_X^{\boxplus\alpha}$ exists for $\boxplus$-infinitely divisible distributions (easier, also thanks to the Popa-Vinnikov characterization); (ii) show that $\mu_X^{\uplus\alpha}$ exists for any cp map α; (iii) Show that $\mu_X^{\boxplus\alpha} = (\mathbb{B}(\mu^{\uplus\alpha}))^{\uplus(\alpha-1)^{-1}\alpha}$, where $\mathbb{B}$ is the *Boolean-to-free Bercovici-Pata bijection*.

Free convolution semigroups indexed by cp maps

As in the scalar-valued context, the Boolean-to-free Bercovici-Pata bijection from the operator-valued context embeds in a semigroups of maps on the space of distributions:

$$\{\mathbb{B}_\alpha(\mu) = \left(\mu^{\boxplus(1+\alpha)}\right)^{\uplus(1+\alpha)^{-1}} : \alpha \colon B \to B \text{ cp map}\}.$$

$\mathbb{B}_\alpha(\mu)$ is $\boxplus$-infinitely divisible for any μ when $\alpha = 1$; we get then the Boolean-to-free Bercovici-Pata bijection. Moreover, again as in the scalar-valued context, the map $h(\alpha, b) = G_{\mathbb{B}_\alpha(\mu)}(b)^{-1} - b$ satisfied a "complex Burgers equation:"

$$\frac{\partial h(\alpha, b)}{\partial \alpha}(\rho) - \frac{\partial h(\alpha, b)}{\partial b}(\rho(h(\alpha, b))) = 0,$$

where ρ, α are cp, and $\Im b > 0$.

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Concluding remarks

- One can generalize many results related to convolution semigroups to the operator-valued context;
- Analytic tools (the fully matricial maps) work in the operator-valued case almost as well as in the scalar valued case;
- The full strength of Voiculescu's subordination result has not been used in the proofs. (So there must be more...)

Thank you!