An application of property (T) for discrete quantum groups

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Compact quantum groups

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Definition

$$\mathbb{G} = \big(\mathrm{C}(\mathbb{G}), \Delta\big)$$

- $\mathrm{C}(\mathbb{G})$ — unital C^*-algebra
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- The span $\mathrm{Pol}(\mathbb{G})$ of matrix elements of all irreducible corepresentations of $\mathbb{G}$ is a Hopf algebra dense in $\mathrm{C}(\mathbb{G})$.

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Fact

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- Existence of exotic norms is interesting for the theory of quantum group actions.

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- Recall: $\boldsymbol{w} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathbb{G})} u^{\alpha} \in \mathrm{M}\big(\mathrm{c}_0(\widehat{\mathbb{G}}) \otimes \mathrm{C}(\mathbb{G}_{\max})\big)$.
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Definition (P. Fima, 2008)

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This provides many examples such that

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(take $\mathbb{G} = \widetilde{\mathbb{G}_{\min}}$ with $\mathbb{G}$ not coamenable and $\widehat{\mathbb{G}}$ without property (T)).

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Fact

Any discrete quantum group with property (T) is unimodular.

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- Denote the resulting quantum group by $\mathbb{G}_{\Pi}$.

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