Gog, Magog and Schützenberger involution

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Bialgebras in Free Probability

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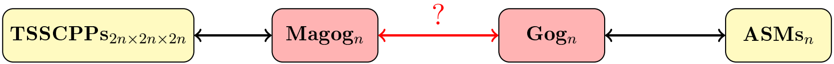
1. Objects
   - Alternating sign matrices
   - TSSCPPs

2. Gelfand-Tsetlin Triangles
   - Gog triangles
     - Statistics
     - Gog Trapezoid
   - Magog triangles
     - Statistics
     - Magog trapezoid

3. A construction
   - Schützenberger involution
   - Treatment of inversions

Scheme


$\mathbf{TSSCPPs}_{2n\times 2n\times 2n}$
$\mathbf{Magog}_n$
?
$\mathbf{Gog}_n$
$\mathbf{ASMs}_n$

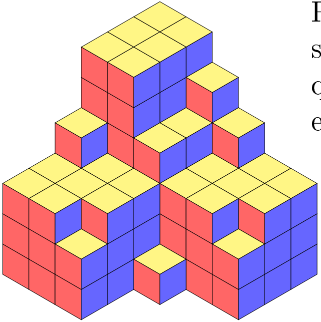
Alternating sign matrices(ASM)

Square matrix of 0s,1s and -1s, for which

- The sum of the entries in each row and in each column is 1.
- The non-zero entries of each row and of each column alternate in sign.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Plane partitions(PP) : can be visualized as a stack of unit cubes pushed into a corner.

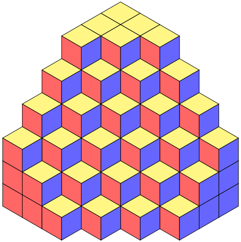


PP is a 3D-partitions : bidimensional tableau h of non increasing sequence of integers in each row and each column

665433
664333
664332
4331
333
332

TSSCPP

Totally Symmetric Self-Complementary Plane Partition (TSSCPP)



TSSCPP : PP with all symmetries of hexagon $\Rrightarrow$ reduced to the fondamental domain Twelfth of the hexagon.

Enumeration and problematic

Theorem [Zeilberger]

$$|\text{ASMs}_{n\times n}| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = |\text{TSSCPPs}_{2n\times 2n\times 2n}|$$

Enumeration and problematic

Theorem [Zeilberger]

$$|\text{ASMs}_{n\times n}| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = |\text{TSSCPPs}_{2n\times 2n\times 2n}|$$

Open problem

Give a bijective construction .

Gelfand-Tsetlin triangle of size n

Triangular array of $n(n+1)/2$ positive integers $X = (x_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$

$$\begin{array}{ccccccccccccc}
x_{n,1} & & \leqslant & & x_{n,2} & & \leqslant & & x_{n,3} & \leqslant & \cdots & \leqslant & x_{n,n} \\
& \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & & & \rotatebox{45}{$\leqslant$} & \\
& & x_{n-1,1} & \leqslant & & & x_{n-1,2} & \leqslant & \cdots & \leqslant & x_{n-1,n-1} & & \\
& & & \rotatebox{-45}{$\leqslant$} & & & & & & \rotatebox{45}{$\leqslant$} & & & \\
& & & & x_{n-2,1} & & \cdots & & x_{n-2,,n-2} & & & & \\
& & & & & \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & & & & \\
& & & & & \cdots & & \cdots & & & & & \\
& & & & & \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & & & & \\
& & & & & & x_{1,1} & & & & & &
\end{array}$$

Gog triangle of size n

Triangular array of $n(n+1)/2$ positive integers $\in [\![1, n]\!]$

$$
\begin{array}{ccccccccc}
x_{n,1} & < & x_{n,2} & < & x_{n,3} & < & \cdots & < & x_{n,n} \\
& \rotatebox{-45}{$\leq$} \; \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} \; \rotatebox{45}{$\leq$} & & & & \rotatebox{45}{$\leq$} & \\
& x_{n-1,1} & < & x_{n-1,2} & < & \cdots & < & x_{n-1,n-1} & \\
& & \rotatebox{-45}{$\leq$} & & & & \rotatebox{45}{$\leq$} & & \\
& & x_{n-2,1} & & \cdots & & x_{n-2,n-2} & & \\
& & & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & & & \\
& & & \cdots & & \cdots & & & \\
& & & & \rotatebox{-45}{$\leq$} \; \rotatebox{45}{$\leq$} & & & & \\
& & & & x_{1,1} & & & &
\end{array}
$$

Gog triangle of size n

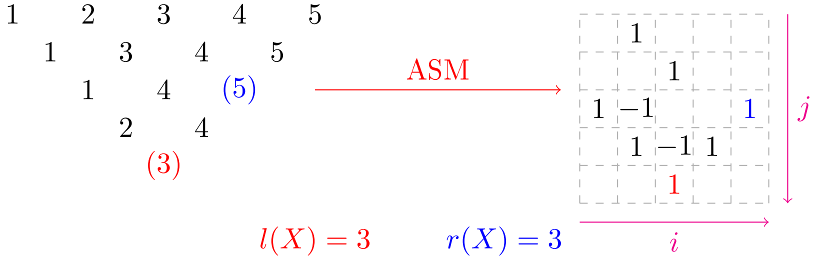
Triangular array of $n(n+1)/2$ positive integers $\in [\![1, n]\!]$

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & \cdots & & n \\
= & & = & & = & & & & = \\
x_{n,1} & < & x_{n,2} & < & x_{n,3} & < & \cdots & < & x_{n,n} \\
\leq & & \leq \quad \leq & & \leq & & & \leq & \\
& x_{n-1,1} & < & x_{n-1,2} & < & \cdots & < & x_{n-1,n-1} & \\
& \leq & & & & & \leq & & \\
& & x_{n-2,1} & & \cdots & & x_{n-2,n-2} & & \\
& & \leq & & & \leq & & & \\
& & & \cdots & & \cdots & & & \\
& & & \leq & \leq & & & & \\
& & & & x_{1,1} & & & &
\end{array}$$

Statistics

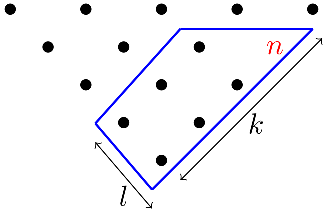
Let X Gog triangle of size n.

- $l(X)$: $x_{1,1}$
- $r(X)$: $\sharp k$ such that $x_{k,k} = n$



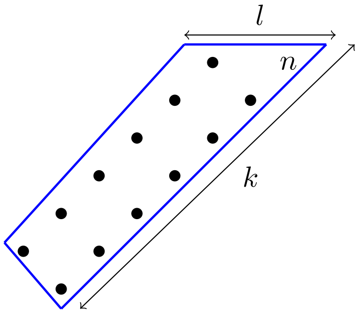
Gog Trapezoid

$(n, k, l)-$Gog trapezoid : The l rightmost SW-NE diagonals of the k bottom rows such that $(l \leqslant k)$ of Gog triangle of height n.



$(n, 4, 2)$-Gog trapezoid

Canonical completion of a $(n, k, l)-$Gog trapezoid to a $(n, k)-$Gog trapezoid by adding on its left the minimal Gog triangle of size $k - l$:



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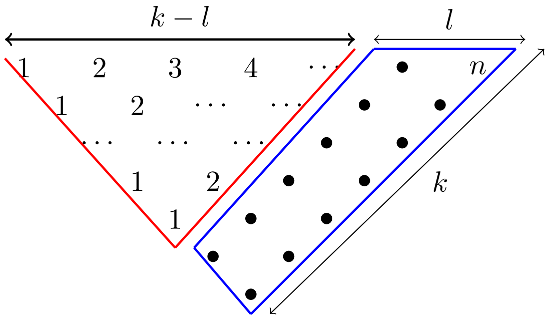

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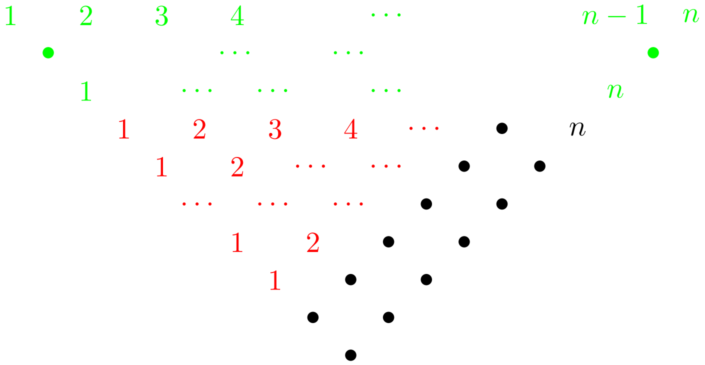

Magog triangles of size n

Triangular array of $n(n+1)/2$ positive integers $\in [\![1, n]\!]$

$$
\begin{array}{ccccccccccc}
x_{n,1} & \leqslant & x_{n,2} & \leqslant & x_{n,3} & \leqslant & \cdots & & x_{n,n} & \leqslant n \\
\rotatebox{-60}{$\leqslant$} & \rotatebox{60}{$\leqslant$} & \rotatebox{-60}{$\leqslant$} & \rotatebox{60}{$\leqslant$} & & & & \rotatebox{60}{$\leqslant$} & & \\
& x_{n-1,1} & \leqslant & x_{n-1,2} & \leqslant & \cdots & \leqslant & x_{n-1,n-1} & \leqslant n-1 & \\
& \rotatebox{-60}{$\leqslant$} & & & & & \rotatebox{60}{$\leqslant$} & & & \\
& & x_{n-2,1} & & \cdots & & x_{n-2,n-2} & \leqslant n-2 & & \\
& & \rotatebox{-60}{$\leqslant$} & & & \rotatebox{60}{$\leqslant$} & & & & \\
& & & \cdots & & \cdots & & & & \\
& & & \rotatebox{-60}{$\leqslant$} & \rotatebox{60}{$\leqslant$} & & & & & \\
& & & & x_{1,1} & \leqslant 1 & & & &
\end{array}
$$

Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
- $r(X)$: The exit path of X

$$\begin{array}{ccccccccccc}
1 & & 1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 1 & & 3 & & 3 & & 4 & \\
& & 1 & & 1 & & 3 & & 4 & & \\
& & & 1 & & 2 & & 3 & & & \\
& & & & 1 & & 2 & & & & \\
& & & & & 1 & & & & &
\end{array}$$

Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
- $r(X)$: The exit path of X

1 1 2 3 4 5 $\xrightarrow{\Sigma}$ 16
1 1 3 3 4 $\xrightarrow{\Sigma}$ 12
1 1 3 4
1 2 3
1 2
1

$l(X) = 4$

Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
- $r(X)$: The exit path of X

1	1	2	3	4	5
1	1	3	3	4	
1	1	3	4 $\longleftarrow x_{kk}$		
1	2	3			
1	2				
1					

$l(X) = 4$

$k = \max\{j / x_{jj} = j\}$

Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
- $r(X)$: The exit path of X

$$
\begin{array}{ccccccccccc}
1 & & 1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 1 & & 3 & & 3 & & 4 & \\
& & 1 & & 1 & & 3 & \longleftarrow & 4 & \longleftarrow & x_{kk} \\
& & & 1 & & 2 & & 3 & & & \\
& & & & 1 & & 2 & & & & \\
& & & & & 1 & & & & &
\end{array}
$$

$$\boxed{l(X) = 4}$$

$$k = \max\{j / x_{jj} = j\}$$

Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
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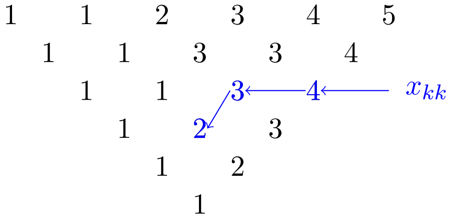

$l(X) = 4$

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Statistics

Let X a Magog triangle of size n

- $l(X) = \left(\sum_{j=1}^{n} x_{n,j} - \sum_{j=1}^{n-1} x_{n-1,j}\right)$
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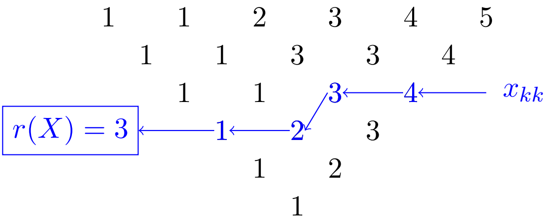


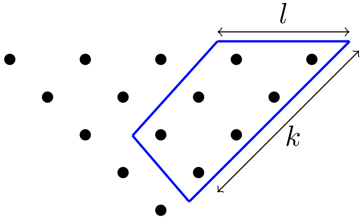
$l(X) = 4$

$k = \max\{j / x_{jj} = j\}$

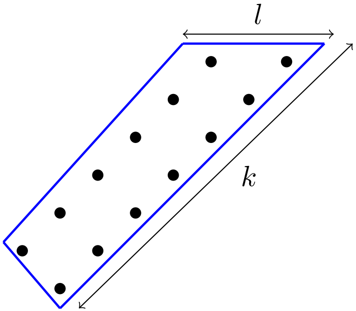
Magog Trapezoid

$(n,k,l)-$Magog trapezoid : the l rightmost SW-NE diagonals of the k top rows such that $(l \leqslant k)$ of Magog triangle of height n.

$$n = \max\{x_{i,i} - i + k, \quad 1 \leqslant i \leqslant k\}$$



Canonical completion of a (n, k, l)−Magog trapezoid to a (n, k)−Magog trapezoid by adding on its left the minimal Magog triangle of size $k - l$:



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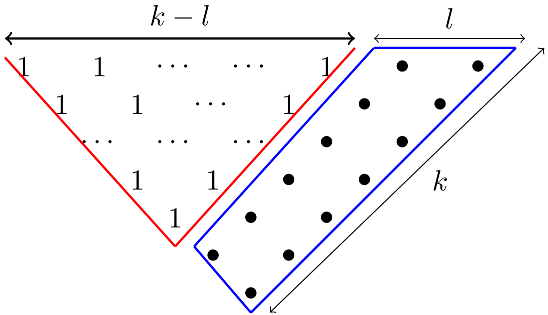

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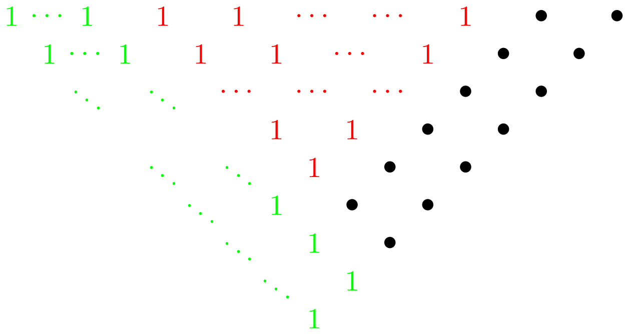


Conjecture on triangles

$$|Gog|_{n,r,l} = |Magog|_{n,r,l}$$

<table>
<tr><th>r \ l</th><th>1</th><th colspan="2">2</th><th colspan="2">3</th></tr>
<tr><td>1</td><td>1 2 3
1 2
1</td><td colspan="2">1 2 3
1 3
1</td><td colspan="2">0</td></tr>
<tr><td>2</td><td>1 2 3
1 2
2</td><td>1 2 3
2 3
2</td><td>1 2 3
1 2
2</td><td colspan="2">0</td></tr>
<tr><td>3</td><td>0</td><td colspan="2">0</td><td>1 2 3
1 3
3</td><td>1 2 3
2 3
3</td></tr>
</table>

Gog triangles of size 3

<table>
<tr><th>r \ l</th><th>1</th><th colspan="2">2</th><th colspan="2">3</th></tr>
<tr><td>1</td><td>1 1 1
1 1
1</td><td colspan="2">1 1 2
1 1
1</td><td colspan="2">0</td></tr>
<tr><td>2</td><td>1 1 2
1 2
1</td><td>1 2 2
1 2
1</td><td>1 1 3
1 2
1</td><td colspan="2">0</td></tr>
<tr><td>3</td><td>0</td><td colspan="2">0</td><td>1 1 3
1 1
1</td><td>1 2 3
1 2
1</td></tr>
</table>

Magog triangles of size 3

Example for $n = 7$

r \ l	1	2	3	4	5	6	7
1	429	1287	2002	2002	1287	429	0
2	1287	4160	6838	7176	4849	1716	0
3	2002	6838	11908	13260	9594	3718	0
4	2002	7176	13260	15912	12714	5720	0
5	1287	4849	9594	12714	11869	7007	0
6	429	1716	3718	5720	7007	7436	0
7	0	0	0	0	0	0	7436

Conjecture on trapezoide

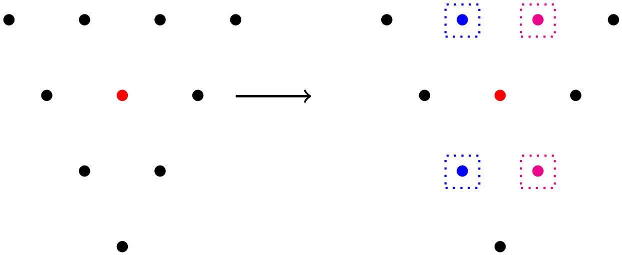
There exists a bijection between Gog and Magog triangles which preserves the statistics r and l and the trapezoids.

We give here a construction of an explicit bijection between $(n, 3)-$Gog trapezoids and $(n, 3)-$Magog trapezoids.
This construction is based on two operations

- Treatment of an inversion φ.
- Schützenberger involution S.

Schützenberger involution

Let a Gelfand-Tsetlin trapezoid of size n . Let us define the operation τ which acts on one element as follows :



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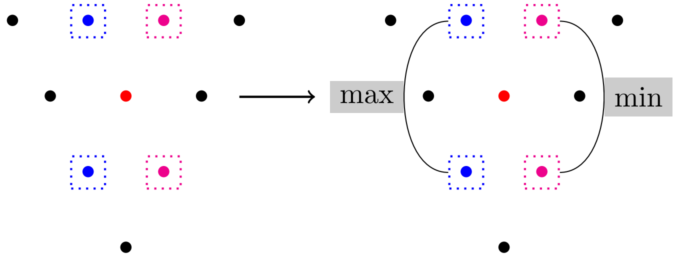


$$\tau(\bullet) = \boxed{\max} + \boxed{\min} - \bullet$$

τ_i acts on all the elements of the i^{th} row.

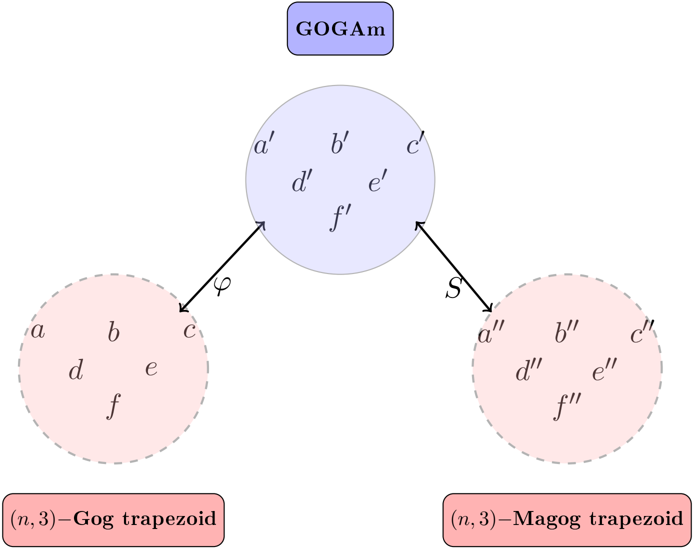
Schützenberger involution

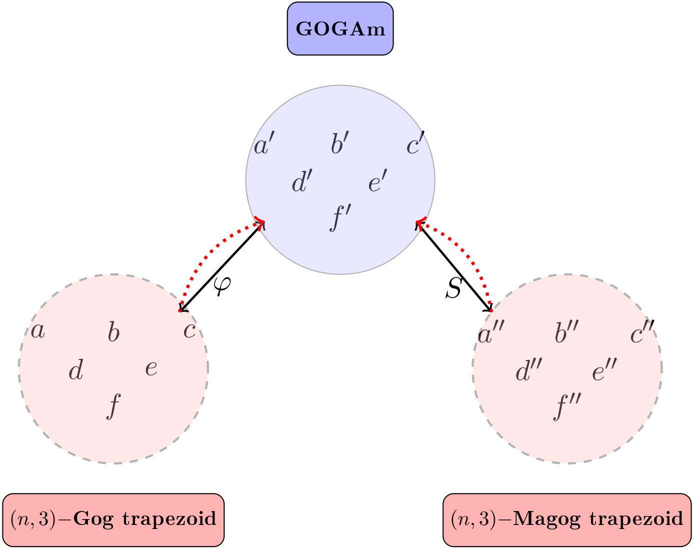
Let a Gog or Magog trapezoid of size n. We denote S_n the Schützenberger involution which acts as follows

$$S_n = \tau_1\tau_2\cdots\tau_{n-1}\tau_1\tau_2\cdots\tau_{n-2}\cdots\tau_1\tau_2\tau_1$$

Example :

$$\begin{array}{ccccc} 1 & & 6 & & 8 \\ & 2 & & 8 & \\ & & 4 & & \end{array} \xrightarrow{\tau_1} \begin{array}{ccccc} 1 & & 6 & & 8 \\ & 2 & & 8 & \\ & & 6 & & \end{array} \xrightarrow{\tau_2} \begin{array}{ccccc} 1 & & 6 & & 8 \\ & 5 & & 6 & \\ & & 6 & & \end{array} \xrightarrow{\tau_1} \begin{array}{ccccc} 1 & & 6 & & 8 \\ & 5 & & 6 & \\ & & 5 & & \end{array}$$


GOGAm
a' b' c'
d' e'
f'
φ
S
a b c
d e
f
a'' b'' c''
d'' e''
f''
$(n,3)-$Gog trapezoid
$(n,3)-$Magog trapezoid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a' b' c'
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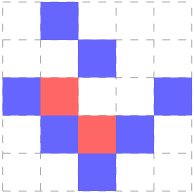
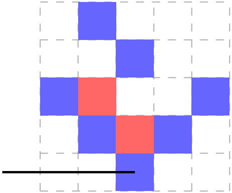
φ
S
a b c
d e
f
a'' b'' c''
d'' e''
f''
$(n,3)-$Gog trapezoid
$(n,3)-$Magog trapezoid

Inversion in Gog triangle



```
1     2     3     4     5
   1     3     4     5
      1     4     5
         2     4
            3
```

Inversion in Gog triangle



$$
\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
& 1 & & 3 & & 4 & & 5 & \\
& & 1 & & 4 & & 5 & & \\
& & & 2 & & 4 & & & \\
& & & & 3 & & & &
\end{array}
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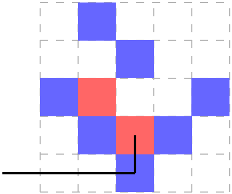
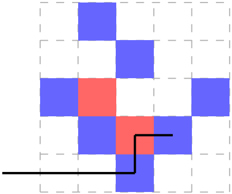


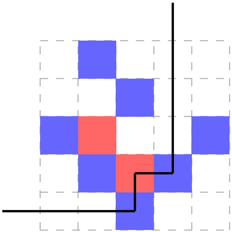
1 2 3 4 5
1 3 4 5
1 4 5
2 4
3

Inversion in Gog triangle



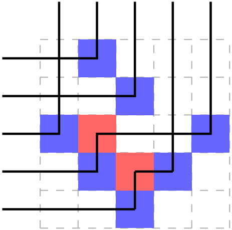
1 2 3 4 5
1 3 4 5
1 4 5
2 4
3

Inversion in Gog triangle



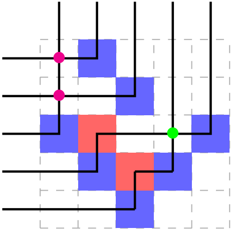
1 2 3 4 5
1 3 4 5
1 4 5
2 4
3

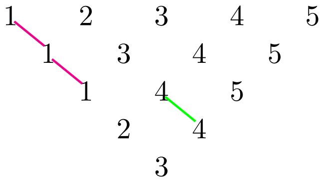
Inversion in Gog triangle



1 2 3 4 5
1 3 4 5
1 4 5
2 4
3

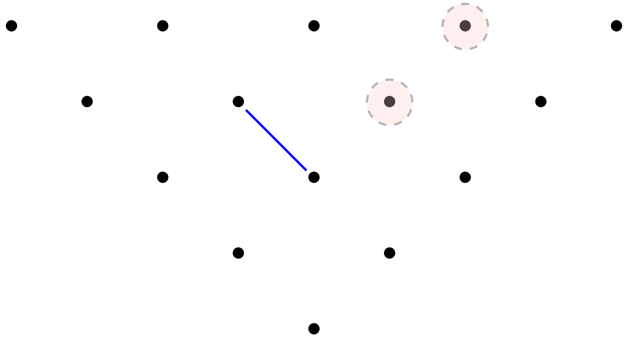
Inversion in Gog triangle





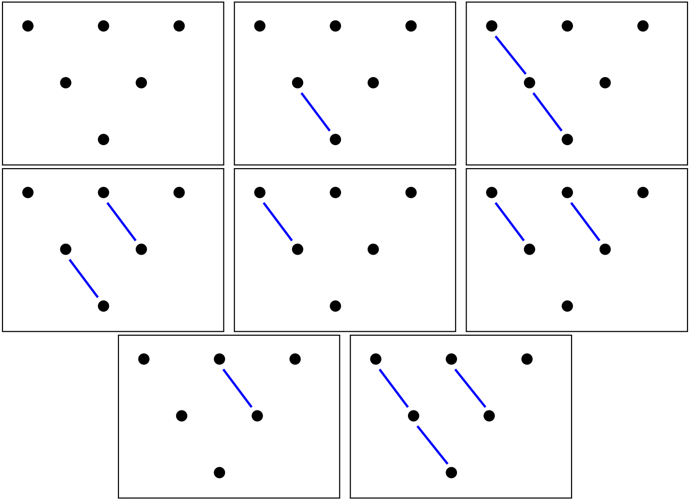
Inversion in triangle

Let x a Gog trapezoid. x contains a inversion



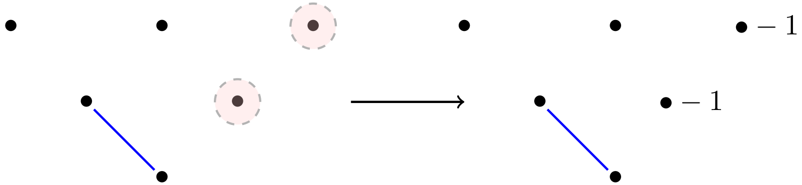
The elements  are covered by the inversion.

Configurations of inversions in $(n,3)-$Gog trapezoids

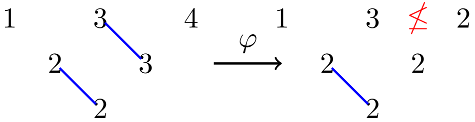


Treatment of inversion (φ)

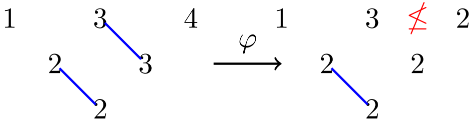
Treament of inversion acts on the $(n, 3)-$Gog trapezoids by substracting 1 to the elements covered by the inversion.



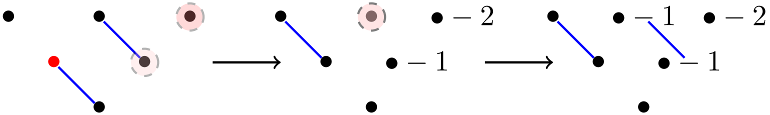
Problem



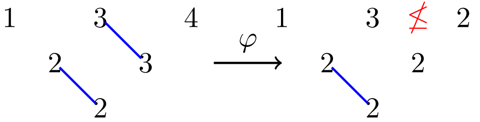
Problem



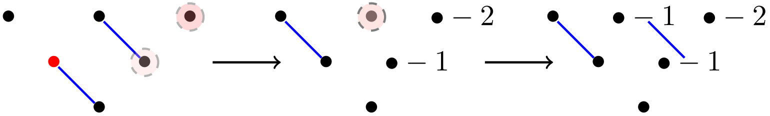
Solution

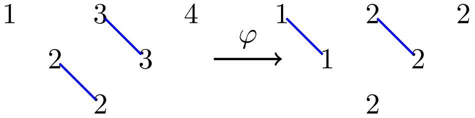


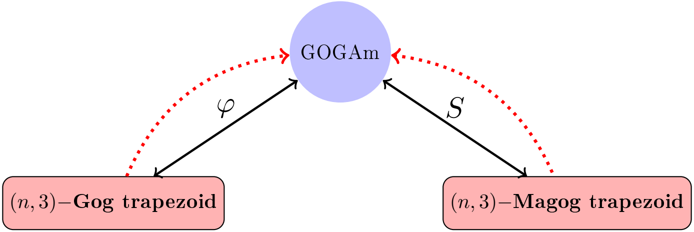
Problem

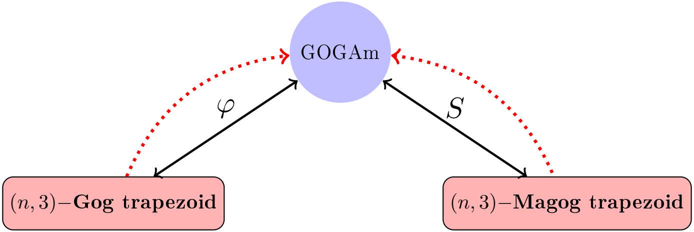


Solution






GOGAm
φ
S
$(n,3)-$**Gog trapezoid**
$(n,3)-$**Magog trapezoid**

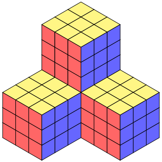


Theorem [C, Biane]

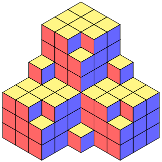
This algorithm is a bijection which preserves the statistics l and r.

Conclusion

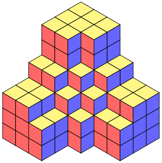
$$\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$



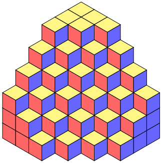
$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$



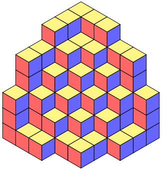
$$\begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$



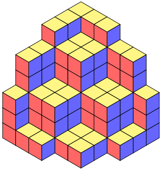
$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$



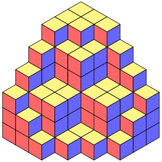
$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$



$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$



Some further rules allow to extend this bijection between $(n, k, 2)-$Gog trapezoid and $(n, k, 2)-$Magog trapezoid