Truncations of Haar distributed matrices and bivariate Brownian bridge

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Joint work with Alain Rouault (Versailles)

G. Chapuy (2007) : σ uniform on S_n. Define for $p, q \leq n$

$$X^{(n)}_{p,q} = \#\{1 \leq i \leq p \ , \ \sigma(i) \leq q\}.$$

A suitable normalisation of the sequence of two-parameters processes $X^{(n)}_{\lfloor ns \rfloor, \lfloor nt \rfloor}$ converge in distribution to the bivariate Brownian bridge.

Note that

$$X^{(n)}_{p,q} = \mathrm{Tr}(\Sigma_{p,q}\Sigma^{\star}_{p,q})$$

where $\Sigma_{p,q}$ is the truncated matrix of size $p \times q$ of Σ, the permutation matrix associated with σ.

Similar result when Σ replaced by a Haar distributed matrix ?

Main result: Let U be a Haar unitary, resp. orthogonal, matrix in $\mathbb{U}(n)$, resp. in $\mathbb{O}(n)$. We consider, for $p \leq n$ and $q \leq n$, the upper-left $p \times q$ submatrix $V_{p,q}$.
Set

$$T_{p,q} = \mathrm{Tr}(V_{p,q}V_{p,q}^\star) = \sum_{i\leq p, j\leq q} |U_{i,j}|^2 \,. \tag{1}$$

and

$$Y_{p,q}^{(n)} = T_{p,q} - \mathbb{E}T_{p,q}.$$

We define a sequence of two-parameter processes $W^{(n)}$ by

$$W^{(n)} := \left(Y^{(n)}_{\lfloor ns\rfloor,\lfloor nt\rfloor}, s,t \in [0,1]\right).$$

$W^{(n)} \in D([0,1]^2)$ the set of functions on $[0,1]^2$ which are right continuous and with limits in all orthants, endowed with the Skorohod topology.

Theorem 1 *The process* $W^{(n)}$ *converges in distribution in* $D([0,1]^2)$ *to a tied-down Brownian bridge* $\sqrt{\frac{2}{\beta}}W^{(\infty)}$ *where* $W^{(\infty)}$ *is a centered continuous Gaussian process on* $[0,1]^2$ *of covariance*

$$\mathbb{E}[W^{(\infty)}(s,t)W^{(\infty)}(s',t')] = (s \wedge s' - ss')(t \wedge t' - tt'),$$

$\beta = 2$ *in the unitary case and* $\beta = 1$ *in the orthogonal case.*

Some related works:

- d'Aristotile, Diaconis and Newman :

$$\left\{ \sum_{i=1}^{\lfloor ns \rfloor} U_{ii} \ , \ s \in [0,1] \right\}_n \overset{law}{\to} MB(\mathbb{C})$$

- Silverstein: q fixed

$$\left\{ n^{1/2}\left(\sum_{i=1}^{\lfloor ns \rfloor} |U_{iq}|^2 - s\right) \ , \ s \in [0,1] \right\}_n \overset{law}{\to} BB$$

The covariance function (unitary case)

From the invariance of the Haar measure, we have:

$$E|U_{i,j}|^2 = \frac{1}{n},\ \mathbb{E}|U_{i,j}|^4 = \frac{2}{n(n+1)}$$

For $j \neq k$,

$$E\left(|U_{i,j}|^2|U_{i,k}|^2\right) = \frac{1}{n(n+1)}$$

and if $i \neq k, j \neq \ell$,

$$E\left(|U_{i,j}|^2|U_{k,\ell}|^2\right) = \frac{1}{n^2-1}\,.$$

From these relations,

$$E(T_{p,q}) = \frac{pq}{n}\,.$$

and

$$Var(T_{p,q}) = pq\frac{n^2 - n(p+q) + pq}{n^2(n^2-1)} \rightarrow_{n\to\infty} s(1-s)t(1-t)$$

if $p/n \to s$, $q/n \to t$.

Proof of Theorem 1:

- tightness of the distributions of $W^{(n)}$ in $D([0,1]^2)$: need to estimate moments of increments of the process $W^{(n)}$ over rectangles.

- convergence of the finite dimensional distributions : for $(s_i, t_i)_{i \leq k} \in [0,1]^2$, prove that

$$(W^{(n)}_{s_i,t_i}, i \leq k) \overset{(law)}{\to} \text{ Gaussian vector,}$$

or equivalently for $(a_i)_{i \leq k} \in \mathbb{R}$,

$$X^{(n)} := \sum_{i=1}^{k} a_i W^{(n)}_{s_i,t_i} \overset{(law)}{\to} N(0, \sum_{i,j} a_i a_j \Gamma((s_i, t_i); (s_j, t_j)))$$

To prove the convergence of $X^{(n)}$, show that $\kappa_r(X^{(n)}) \to 0$ for $r \geq 3$ where κ_r denotes the cumulant of order r.

Weingarten functions (Collins)

a) *Unitary case :* for any choice of indices i_l, j_m in $\{1, \ldots, n\}$,

$$\mathbb{E}\left(U_{i_1 j_1} \ldots U_{i_k j_k} \bar{U}_{i_{\bar{1}} j_{\bar{1}}} \ldots \bar{U}_{i_{\bar{k}}, j_{\bar{k}}}\right) = \sum_{p_1, p_2 \in \mathcal{M}_{2k}^U} \delta_{\mathbf{i}}^{p_1} \delta_{\mathbf{j}}^{p_2} \, \mathrm{Wg}^{\mathbb{U}(n)}(p_1, p_2)$$

where $\mathcal{M}_{2k}^U$ is the set of pairings of $[2k] = \{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}$ pairing one element of $[k]$ and one element of $[\bar{k}]$.
$\mathrm{Wg}^{\mathbb{U}(n)}$ is the Weingarten matrix , the pseudo inverse of a Gram matrix.

b) *Orthogonal case:*

$$\mathbb{E}\left(O_{i_1 j_1} \ldots O_{i_k j_k} O_{i_{\bar{1}} j_{\bar{1}}} \ldots O_{i_{\bar{k}} j_{\bar{k}}}\right) = \sum_{p_1, p_2 \in \mathcal{M}_{2k}} \delta_{\mathbf{i}}^{p_1} \delta_{\mathbf{j}}^{p_2} \, \mathrm{Wg}^{\mathbb{O}(n)}(p_1, p_2)$$

where $\mathcal{M}_{2k}$ is the set of pairings of $[2k] = \{1, \ldots, k, \bar{1}, \ldots, \bar{k}\}$, $\mathrm{Wg}^{\mathbb{O}(n)}$ is the orthogonal Weingarten matrix.

Another formulation: unitary case

$$\mathbb{E}\left(U_{i_1 j_1} \ldots U_{i_k, j_k} \bar{U}_{i_{\bar{1}} j_{\bar{1}}} \ldots \bar{U}_{i_{\bar{k}}, j_{\bar{k}}}\right) = \sum_{\alpha, \beta \in \mathcal{S}_k} \tilde{\delta}_{\mathbf{i}}^{\alpha} \tilde{\delta}_{\mathbf{j}}^{\beta} \, \mathrm{Wg}(n, \beta\alpha^{-1})$$

where $\tilde{\delta}_{\mathbf{i}}^{\alpha} = 1$ if $i(s) = i(\overline{\alpha(s)})$ for every $s \leq k$ and 0 otherwise.
In particular, if $\pi \in \mathcal{S}_k$,

$$\mathrm{Wg}(n, \pi) = \mathbb{E}(U_{11} \ldots U_{kk} \bar{U}_{1\pi(1)} \ldots \bar{U}_{k\pi(k)}) \, .$$

$\mathcal{S}_k$ is the set of permutations of $[k]$.

Convergence of the finite dimensional distributions
Note that $T_{p,q} = \mathrm{Tr}(D_1 U D_2 U^\star)$ where $D_1 = diag(I_p, 0_{n-p})$, $D_2 = diag(I_q, 0_{n-q})$.
The asymptotic vanishing cumulants of $X^{(n)} := \sum_{i=1}^k a_i W^{(n)}_{s_i,t_i}$ follows from:

Proposition 1 *Unitary case (Mingo, Sniady, Speicher)*
Let $D = (D_1, \dots D_k)$ and $\bar{D} = (D_{\bar{1}}, \dots D_{\bar{k}})$ be two families of deterministic matrices of size n. We set, for $1 \le i \le r$,

$$X_i = \mathrm{Tr}(D_i U D_{\bar{i}} U^\star)\,.$$

Then,

$$\kappa_r(X_1, \dots, X_r) = \sum_{\alpha,\beta \in \mathcal{S}_r} \sum_{A} C_{\beta\alpha^{-1},A}\, \mathrm{Tr}_\alpha(\bar{D})\, \mathrm{Tr}_{\beta^{-1}}(D) \qquad (2)$$

where in the second sum, $A \in \mathcal{P}(r)$ is such that

$$\beta\alpha^{-1} \le A \text{ and } A \vee \beta \vee \alpha = 1_r\,, \qquad (3)$$

and $C_{\sigma,A}$ are the relative cumulants of the unitary Weingarten function.

Moreover, if the sequence $\{D, \bar{D}\}_n$ *has a limit distribution, then for* $r \geq 3$,

$$\lim_{n \to \infty} \kappa_r(X_1, \dots, X_r) = 0.$$

Asymptotics:

$$\begin{cases} C_{\sigma,A} = O(n^{-2r-\#(\sigma)+2\#(A)}) \\ \mathrm{Tr}_\alpha(\bar{D}) = O(n^{\#(\alpha)}),\ \mathrm{Tr}_{\beta^{-1}}(D) = O(n^{\#(\beta)}) \\ \beta\alpha^{-1} \leq A \text{ and } A \vee \beta \vee \alpha = 1_r \end{cases}$$

$$\Longrightarrow \kappa_r = O(n^{2-r})$$

From the proposition, we easily get

$$\kappa_r(X^{(n)}) \to 0,\ r \geq 3.$$

The orthogonal case: Description of $\mathcal{M}_{2k}$
For $g \in \mathcal{S}_{2k}$, we define

$$\eta(g) = \prod_{i=1}^{k} (g(i)\ g(\bar{i})) \in \mathcal{M}_{2k}\ .$$

$$H_k = \{g \in \mathcal{S}_{2k};\ g\gamma = \gamma g\} \text{ où } \gamma = \prod_{i=1}^{k} (i\bar{i})$$

$$\eta(g) = \eta(g') \Longleftrightarrow \exists h \in H, g = g'h.$$

$$\mathcal{M}_{2k} \sim \mathcal{S}_{2k}/H_k$$

$$\mathbb{E}\left(O_{i_1 j_1} \dots O_{i_k j_k} O_{i_{\bar{1}} j_{\bar{1}}} \dots O_{i_{\bar{k}} j_{\bar{k}}}\right) = \frac{1}{|H_k|^2} \sum_{g_1, g_2 \in \mathcal{S}_{2k}} \delta_{\mathbf{i}}^{\eta(g_1)} \delta_{\mathbf{j}}^{\eta(g_2)} \operatorname{Wg}(g_1^{-1} g_2)$$

where Wg is invariant on the $H_k \backslash \mathcal{S}_{2k} / H_k$.

Parametrisation of $\mathcal{S}_{2k}/H_k$:
For $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{-1, 1\}^k$, we define τ_ϵ by

$$\tau_\epsilon = \prod_{i, \epsilon_i = -1} (i\bar{i}) \in H_k.$$

For $\pi \in \mathcal{S}_k$, t_π est défini par:

$$t_\pi(i) = i; \; t_\pi(\bar{i}) = \overline{\pi(i)}\,.$$

Each class of $\mathcal{S}_{2k}/H_k$ contains exactely one permutation

$$g_{\epsilon,\pi} = \tau_\epsilon t_\pi$$

for a particular pair (ϵ, π).

Proposition 2 *Let* $D = (D_1, \dots D_k)$ *and* $\bar{D} = (D_{\bar{1}}, \dots D_{\bar{k}})$ *be two families of deterministic and* **symmetric** *matrices of size* n. *We set, for* $1 \leq i \leq r$,

$$X_i = \mathrm{Tr}(D_i O D_{\bar{i}} O^\star)\,.$$

Then,

$$\kappa_r(X_1, \dots, X_r) = \sum_{\alpha,\beta\in\mathcal{S}_r,\epsilon\in\{-1,1\}^r} \lambda_{\alpha,\beta,r} \sum_A C_{\sigma,A}\, \mathrm{Tr}_\alpha(\bar{D})\, \mathrm{Tr}_{\beta^{-1}}(D)$$

where

- $\sigma \in \mathcal{S}_r$ *is a function of* α, β, ϵ *satisfying* $t_{\alpha^{-1}} \tau_\epsilon t_\beta \sim t_\sigma$ in $H_k \backslash S_{2k} / H_k$
- *in the second sum,* $A \in \mathcal{P}(r)$ *is such that*

$$\sigma \leq A \text{ and } A \vee \beta \vee \alpha = 1_r\,, \tag{4}$$

- $C_{\sigma,A}$ *are the relative cumulants of the orthogonal Weingarten function.*

Asymptotics

$$\begin{cases} C_{\sigma,A} = O(n^{-2r-\#(\sigma)+2\#(A)}) \\ \mathrm{Tr}_\alpha(\bar{D}) = O(n^{\#(\alpha)}),\ \mathrm{Tr}_{\beta^{-1}}(D) = O(n^{\#(\beta)}) \\ t_{\alpha^{-1}}\tau_\epsilon t_\beta = \tau_{\epsilon'} t_\sigma h,\ h \in H_k \\ \sigma \leq A \text{ and } A \vee \beta \vee \alpha = 1_r \end{cases}$$

$$\Longrightarrow \kappa_r = O(n^{-1})$$

Lemma 1 *For $A, B \in \mathcal{P}(r)$ we have*

$$\#(A) + \#(B) \leq r + \#(A \vee B)\,.$$

Moreover, if there exists a block A_i of A and B_j of B such that $\#(A_i \cap B_j) = l$, then,

$$\#(A) + \#(B) \leq r - l + 1 + \#(A \vee B)\,.$$

Therefore,

$$\#(\alpha) + \#(\beta) \leq r + \#(\alpha \vee \beta)\,, \tag{5}$$

$$\#(A) + \#(\alpha \vee \beta) \leq r + 1\,, \tag{6}$$

$$\#(A) \leq \#(\sigma)\,. \tag{7}$$

It is enough to study the cases (worst cases):
- $\#(\alpha) + \#(\beta) = r + \#(\alpha \vee \beta)$
- $\#(\alpha) + \#(\beta) = r - 1 + \#(\alpha \vee \beta)$

For example, if i is a fixed point of α and β, then i is a fixed point of σ and in this case (since $A \vee \alpha \vee \beta = 1_r$),

$$\#(A) < \#(\sigma).$$