Path cocycles in quantum Cayley trees and L^2-cohomology

Roland Vergnioux

Université de Caen Basse-Normandie

Wien, april 12th, 2011

Outline

Universal discrete quantum groups

Consider the unital $*$-algebras defined by generators and relations:

$$\mathscr{A}_u(n) = \langle u_{ij} \mid (u_{ij}) \text{ and } (u_{ij}^*) \text{ unitary}\rangle,$$
$$\mathscr{A}_o(n) = \langle u_{ij} \mid u_{ij} = u_{ij}^*, \ (u_{ij}) \text{ unitary}\rangle,$$

with $1 \le i, j \le n$. They become Hopf $*$-algebras with

$$\Delta(u_{ij}) = \sum u_{ik} \otimes u_{kj}, \quad S(u_{ij}) = u_{ji}^*, \quad \epsilon(u_{ij}) = \delta_{ij}.$$

Moreover there exists a unique positive Haar integral $h : \mathscr{A} \to \mathbb{C}$.
We can consider the GNS construction:

$$H = L^2(\mathscr{A}, h), \quad \lambda : \mathscr{A} \to B(H), \quad M = \lambda(\mathscr{A})'' \subset B(H).$$

Classical counterpart: $\mathscr{A} = \mathbb{C}G$, $H = \ell^2(G)$, with G a discrete group.
Heuristically : $\mathscr{A}_u(n) = \mathbb{C}FU_n$, $\mathscr{A}_o(n) = \mathbb{C}FO_n$, where FO_n, FU_n are discrete quantum groups.

Analogies with free group algebras

- there are natural maps $\mathscr{A}_u(n) \twoheadrightarrow \mathbb{C}F_n$, $\mathscr{A}_o(n) \twoheadrightarrow \mathbb{C}(\mathbb{Z}/2\mathbb{Z})^{*n}$;
- we have $\mathscr{A}_u(n) \twoheadrightarrow \mathscr{B}$ for any $\mathscr{B}$ associated with a unimodular discrete quantum group and some n ;
- FU_n, FO_n have the Property of Rapid Decay ;
- the C^*-algebras $A_u(n)_{\mathrm{red}}$, $A_o(n)_{\mathrm{red}}$ are simple, non-nuclear, exact ;
- $M = \lambda(\mathscr{A}_o(n))'', \lambda(\mathscr{A}_u(n))''$ are solid II_1 factors ;
- FO_n is K-amenable ;
- FO_n, FU_n satisfy Haagerup's Approximation Property.

[Banica, V., Vaes, Vander Vennet, Voigt, Brannan]

The case $n = 2$ behaves differently, e.g. $\mathscr{A}_o(I_2) = \mathscr{C}(SU_{-1}(2))$ has polynomial growth, and will be excluded in this talk.

The main result

For an ICC group G, we can take $\mathscr{A} = \mathbb{C}G$ and consider the Hochschild cohomology groups $H^1(\mathscr{A}, {}_\lambda H_\epsilon)$ and $H^1(\mathscr{A}, {}_\lambda M_\epsilon)$.
These groups are moreover right M-modules and we have

$$\beta_1^{(2)}(G) = \dim_M H^1(\mathscr{A}, H) = \dim_M H^1(\mathscr{A}, M).$$

Recall that $\beta_1^{(2)}(F_n) = n-1$. In the case of the orthogonal universal discrete quantum groups we have the strongly contrasting result:

Theorem

For $n \geq 3$ we have $H^1(\mathscr{A}_o(n), H) = H^1(\mathscr{A}_o(n), M) = 0$.
In particular $\beta_1^{(2)}(\mathscr{A}_o(n)) = 0$. On the other hand $\beta_1^{(2)}(\mathscr{A}_u(n)) \neq 0$.

The main result

Consider a representation $\pi : \mathscr{A} \to L(X)$ on a vector space X.
A π-cocycle is a map $c : \mathscr{A} \to X$ such that

$$\forall x, y \in \mathscr{A} \qquad c(xy) = \pi(x)c(y) + c(x)\epsilon(y).$$

It is trivial if $c(x) = \pi(x)\xi - \xi\epsilon(x)$ for some $\xi \in X$ and all $x \in \mathscr{A}$.
$H^1(\mathscr{A}, X)$ is the space of π-cocycles modulo trivial cocycles.

Theorem

For $n \geq 3$ we have $H^1(\mathscr{A}_o(n), H) = H^1(\mathscr{A}_o(n), M) = 0$.
In particular $\beta_1^{(2)}(\mathscr{A}_o(n)) = 0$. On the other hand $\beta_1^{(2)}(\mathscr{A}_u(n)) \neq 0$.

The main result

Theorem

For $n \geq 3$ we have $H^1(\mathscr{A}_o(n), H) = H^1(\mathscr{A}_o(n), M) = 0$.
In particular $\beta_1^{(2)}(\mathscr{A}_o(n)) = 0$. On the other hand $\beta_1^{(2)}(\mathscr{A}_u(n)) \neq 0$.

Remarks:

- Collins-Härtel-Thom: $\beta_k^{(2)}(\mathscr{A}_o(n)) = 0$ for $k \geq 4$, $\beta_k^{(2)}(A_o(n)) = \beta_{4-k}^{(2)}(A_o(n))$, and Kyed: $\beta_0^{(2)}(\mathscr{A}_o(n)) = 0$.
- Corollary : $\delta^*(\mathscr{A}_o) \leq 1$, by [Connes-Shlyakhtenko]. $\delta(\mathscr{A}_o) = \delta^*(\mathscr{A}_o) = 1$ if M is embeddable into R^ω.
- History : Leuven 11/2008, ArXiv v1 05/2009, ArXiv v2 03/2010

The main result

Theorem

For $n \geq 3$ we have $H^1(\mathscr{A}_o(n), H) = H^1(\mathscr{A}_o(n), M) = 0$.
In particular $\beta_1^{(2)}(\mathscr{A}_o(n)) = 0$. On the other hand $\beta_1^{(2)}(\mathscr{A}_u(n)) \neq 0$.

Strategy (for $\mathscr{A}_o$):

- Show that one particular cocycle vanishes: the *path cocycle* $c_g : \mathscr{A} \to K_g$ with values in the *quantum Cayley tree*
- Prove that this cocycle is "sufficiently universal" and vanishes "sufficiently strongly" (and use Property RD)

More on the strategy

Algebraic version

$$\begin{tikzcd} & X \\ \mathscr{A} \arrow[ur, "c"] \arrow[loop left, "c_0"] & \\ & \mathscr{A}\otimes\mathscr{A} \arrow[ul, "T"] \end{tikzcd}$$

$T = m - (\mathrm{id}\otimes\epsilon)$ is the usual boundary map,
$c_0 = \mathrm{id} - 1\epsilon$ is “the” trivial cocycle on $\mathscr{A}$.
Fix a cocycle $c : \mathscr{A} \to X$.

More on the strategy

Algebraic version

$$\begin{tikzcd}
 & X \\
\mathscr{A} \arrow[ur, "c"] \arrow[loop left, "c_0"] & \\
 & \mathscr{A}\otimes\mathscr{A} \arrow[ul, "T"] \arrow[uu, "m_c"']
\end{tikzcd}$$

Fix a cocycle $c : \mathscr{A} \to X$.
Observe that the cocycle relation for $c : \mathscr{A} \to X$ reads

$$\pi(x)c(y) = c((m - \mathrm{id}\otimes\epsilon)(x\otimes y)) = (c \circ T)(x\otimes y)$$

Define $m_c : \mathscr{A}\otimes\mathscr{A} \to X$ by putting $m_c(x\otimes y) = \pi(x)c(y)$.

More on the strategy

Algebraic version

$$
\begin{tikzcd}
 & X \\
\mathscr{A} \arrow[loop left, "c_0"] \arrow[ur, "c"] \arrow[dr, shift left, "c_g"] & \\
 & \mathscr{A}\otimes\mathscr{A} \arrow[ul, shift left, "T"] \arrow[uu, "m_c"']
\end{tikzcd}
$$

Fix a cocycle $c : \mathscr{A} \to X$.
Define $m_c : \mathscr{A}\otimes\mathscr{A} \to X$ by putting $m_c(x\otimes y) = \pi(x)c(y)$.
Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$, ie $T \circ c_g = c_0$.

We obtain $c = m_c \circ c_g$.

More on the strategy

Algebraic version

$$
\begin{tikzcd}
 & X \\
\mathscr{A} \arrow[ur, "c"] \arrow[dr, shift left, "c_g"] \arrow[loop left, "c_0"] & \\
 & \mathscr{A}\otimes\mathscr{A} \arrow[ul, shift left, "T"] \arrow[uu, "m_c"']
\end{tikzcd}
$$

Fix a cocycle $c : \mathscr{A} \to X$.
Define $m_c : \mathscr{A}\otimes\mathscr{A} \to X$ by putting $m_c(x\otimes y) = \pi(x)c(y)$.
Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to \mathscr{A}\otimes\mathscr{A}$, ie $T \circ c_g = c_0$.

We obtain $c = m_c \circ c_g$. Hence if c_g is trivial with fixed vector $\xi_g \in \mathscr{A}\otimes\mathscr{A}$, all cocycles c are trivial with fixed vector $\xi = m_c(\xi_g)$.

More on the strategy

Hilbertian version

$$\begin{tikzcd}
 & M \\
\mathscr{A} \arrow[ur, "c"] \arrow[dr, shift left, "c_g"] \arrow[loop left, "c_O"] & \\
 & H\otimes H_1 \arrow[uu, "m_c"'] \arrow[ul, shift left, "T"]
\end{tikzcd}$$

Fix a cocycle $c : \mathscr{A} \to M$.
Define $m_c : H\otimes H_1 \to X$ by putting $m_c(x\otimes y) = \pi(x)c(y)$.
Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to H\otimes H_1$, ie $T \circ c_g = c_0$.

We obtain $c = m_c \circ c_g$. Hence if c_g is trivial with fixed vector $\xi_g \in H\otimes H_1$, all cocycles c are trivial with fixed vector $\xi = m_c(\xi_g)$.

With $H_1 \subset H$ finite-dimensional...

More on the strategy

The correct version

$$\begin{tikzcd} & M \\ \mathscr{A} \arrow[ur, "c"] \arrow[loop left, "c_0"] \arrow[dr, shift left, "c_g"] & \\ & H\otimes H_1 \arrow[ul, shift left, "T"] \arrow[uu, "m_c"'] \end{tikzcd}$$

Fix a cocycle $c : \mathscr{A} \to M$.
Define $m_c : H\otimes H_1 \to X$ by putting $m_c(x\otimes y) = \pi(x)c(y)$.
Assume we can "lift" c_0 to a cocycle $c_g : \mathscr{A} \to \mathscr{K}_g'$, ie $T \circ c_g = c_0$.

We obtain $c = m_c \circ c_g$. Hence if c_g is trivial with fixed vector $\xi_g \in M\otimes H_1$, all cocycles c are trivial with fixed vector $\xi = m_c(\xi_g)$.

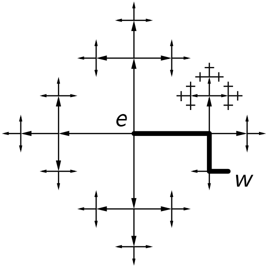
With $\mathscr{K}_g' \subset (\mathscr{A}\otimes\mathscr{A}_1) \oplus \overline{\mathrm{Ker}\, T}$...

Outline

The classical picture

In the case of the free group $F_n = \langle S \rangle$, consider:

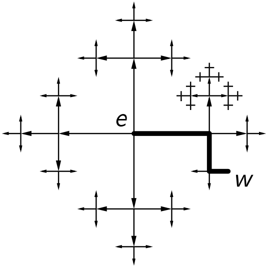
- the Cayley graph of (F_n, S),
- the ℓ^2-space K_g of antisymmetric edges,
- for $w \in F_n$, the sum $c_g(w) \in K_g$ of edges along the path from e to w: "path cocycle"



The classical picture

In the case of the free group $F_n = \langle S \rangle$, consider:

- the Cayley graph of (F_n, S),
- the ℓ^2-space K_g of antisymmetric edges,
- for $w \in F_n$, the sum $c_g(w) \in K_g$ of edges along the path from e to w: "path cocycle"



This cocycle is non-trivial, even proper $\Longrightarrow$ a-T-menability.
In the case of $\mathcal{A}_o(n)$, it will be bounded!

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Cayley graph associated with (G, S) is given by:

- the set of vertices G,
- the set of edges $G \times S$,
- the target map $t : (\alpha, \gamma) \to \alpha\gamma$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the set of vertices G,
- the set of edges $G \times S$,
- the target map $t : (\alpha, \gamma) \to \alpha\gamma$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the set of edges $G \times S$,
- the target map $t : (\alpha, \gamma) \to \alpha\gamma$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target map $t : (\alpha, \gamma) \to \alpha\gamma$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing map $\theta(\alpha, \gamma) = (\alpha\gamma, \gamma^{-1})$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing operator $\Theta : \delta_\alpha \otimes \delta_\gamma \mapsto \delta_{\alpha\gamma} \otimes \delta_{\gamma^{-1}}$.

The quantum Cayley graph

Fix the following data:

- a discrete group G,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing operator $\Theta : \delta_\alpha \otimes \delta_\gamma \mapsto \delta_{\alpha\gamma} \otimes \delta_{\gamma^{-1}}$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

The quantum Cayley graph

Fix the following data:

- a discrete quantum group,
- a finite subset $S \subset G$ such that $S^{-1} = S$, $e \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing operator $\Theta : \delta_\alpha \otimes \delta_\gamma \mapsto \delta_{\alpha\gamma} \otimes \delta_{\gamma^{-1}}$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

The quantum Cayley graph

Fix the following data:

- a discrete quantum group,
- a finite subset $S \subset \mathrm{Irr}\,\mathscr{C}$ such that $\bar{S} = S$ and $\hat{\epsilon} \notin S$.

The Hilbertian Cayley graph associated with (G, S) is given by:

- the space of vertices $H = \ell^2(G)$,
- the space of edges $K = H \otimes H_1$, where $H_1 = \ell^2(S)$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing operator $\Theta : \delta_\alpha \otimes \delta_\gamma \mapsto \delta_{\alpha\gamma} \otimes \delta_{\gamma^{-1}}$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

The quantum Cayley graph

Fix the following data:

- a discrete quantum group,
- a finite subset $S \subset \mathrm{Irr}\,\mathscr{C}$ such that $\bar{S} = S$ and $\hat{\epsilon} \notin S$.

The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

- the space of vertices H,
- the space of edges $K = H \otimes H_1$, where $H_1 = p_S H$,
- the target operator $T : \delta_\alpha \otimes \delta_\beta \mapsto \delta_{\alpha\beta}$,
- the reversing operator $\Theta : \delta_\alpha \otimes \delta_\gamma \mapsto \delta_{\alpha\gamma} \otimes \delta_{\gamma^{-1}}$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

The quantum Cayley graph

Fix the following data:

- a discrete quantum group,
- a finite subset $S \subset \mathrm{Irr}\,\mathscr{C}$ such that $\bar{S} = S$ and $\hat{\epsilon} \notin S$.

The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

- the space of vertices H,
- the space of edges $K = H \otimes H_1$, where $H_1 = p_S H$,
- the target operator $T = m : K \to H$,
- the reversing operator $\Theta = \cdots$, such that $T\Theta = \mathrm{id} \otimes \epsilon$.

$C^*_{\mathrm{red}}(G)$ acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

The quantum Cayley graph

Fix the following data:

- a discrete quantum group,
- a finite subset $S \subset \mathrm{Irr}\,\mathscr{C}$ such that $\bar{S} = S$ and $\hat{\epsilon} \notin S$.

The quantum Cayley graph associated with $(\mathscr{C}, S)$ is given by:

- the space of vertices H,
- the space of edges $K = H \otimes H_1$, where $H_1 = p_S H$,
- the target operator $T = m : K \to H$,
- the reversing operator $\Theta = \cdots$, such that $T\Theta = \mathrm{id} \otimes \epsilon$.

A_{red} acts on H and on the first factor of $H \otimes p_1 H$.
T and Θ are intertwining operators.

Classical subgraphs

Quasi-classical subgraph $Q_0K \subset K$: maximal subspace on which $\Theta^2 = \mathrm{id}$.
Classical subgraph $q_0K \subset Q_0K$: fixed points for the adjoint repr. of $\hat{A}$.

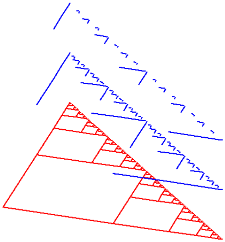
q_0K, Q_0K are not stable under the action of $\mathscr{A}$!

When $\mathscr{A} = \mathbb{C}G$, $q_0K = Q_0K = K$.
When $\mathscr{A} = \mathscr{A}_o(n)$, $q_0K = Q_0K \neq K$.
When $\mathscr{A} = \mathscr{A}_u(n)$, $q_0K \neq Q_0K \neq K$.

Classical subgraphs

Quasi-classical subgraph $Q_0K \subset K$: maximal subspace on which $\Theta^2 = \mathrm{id}$.
Classical subgraph $q_0K \subset Q_0K$: fixed points for the adjoint repr. of $\hat{A}$.

Case of $\mathscr{A}_u(n)$: $q_0K \subset Q_0K \subset K$.

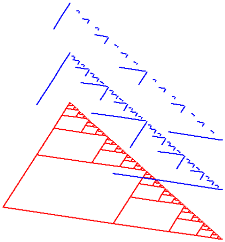


- edge α in the "real" graph $\twoheadrightarrow$ normed vector $\xi_\alpha \in q_0K$
- vertex v in the "real" graph $\twoheadrightarrow$ normed vector $\zeta_v \in q_0H$
- operators T, Θ induced by the classical operations
- but T has weights...

Classical subgraphs

Quasi-classical subgraph $Q_0K \subset K$: maximal subspace on which $\Theta^2 = \mathrm{id}$.
Classical subgraph $q_0K \subset Q_0K$: fixed points for the adjoint repr. of $\hat{A}$.

Case of $\mathscr{A}_u(n)$: $q_0K \subset Q_0K \subset K$.



Why this picture ?

- vertices v are irreducible corepresentations
- edges come from fusion rules $v \otimes u = \bigoplus v'$
- weights depend on dimensions of corepresentations

Quantum path cocycles

We look for cocycles with values in the space of *geometric*, or antisymmetric, edges $K_g = \mathrm{Ker}(\Theta + \mathrm{id})$.
Recall that $T = m = (\mathrm{id} \otimes \epsilon)\Theta$, so that $m - \mathrm{id} \otimes \epsilon = 2T$ on K_g.

Definition

A *path cocycle* is a cocycle $c_g : \mathscr{A} \to K_g$ such that $T \circ c_g = c_0$.

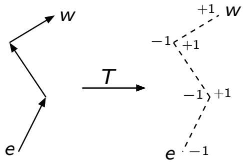
Quantum path cocycles

We look for cocycles with values in the space of *geometric*, or antisymmetric, edges $K_g = \mathrm{Ker}(\Theta + \mathrm{id})$.
Recall that $T = m = (\mathrm{id} \otimes \epsilon)\Theta$, so that $m - \mathrm{id} \otimes \epsilon = 2T$ on K_g.

Definition

A *path cocycle* is a cocycle $c_g : \mathscr{A} \to K_g$ such that $T \circ c_g = c_0$.

Example: in the Cayley tree of F_n, denote by $c_g(w) \in K_g$ the sum of the antisymmetric edges on the path from the origin to w.



Some general results

We consider a free product of A_o's and A_u's with $n \geq 3$.
We denote $\mathscr{K}_g'$ the orthogonal projection of $\mathscr{A} \otimes \mathscr{A}$ onto K_g.

Proposition

If T is injective on $\mathscr{K}_g'$, then there exists a unique path cocycle $c_g : \mathscr{A} \to \mathscr{K}_g'$.

In the case of F_n we have $\mathscr{K}_g' = K_g \cap (\mathscr{A} \otimes \mathscr{A})$ and T is injective only on this dense subspace. On the "purely quantum part" of our quantum trees we have the much stronger property:

Theorem

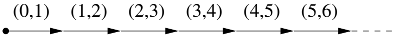
T is injective with closed range on $(1 - Q_0)K_g$.

The orthogonal case

Proposition

In the case of $A_o(Q)$, with $Q \in GL(n, \mathbb{C})$, $Q\bar{Q} \in \mathbb{C}I_n$, $n \geq 3$, the target operator $T : K_g \to H$ is invertible. As a result there exists a unique path cocycle $c_g : \mathscr{A} \to K_g$, and it is trivial.

The main reason is that $q_0 K_g = Q_0 K_g$ comes from the half-line:



We can even compute the fixed vector $\xi_g = T^{-1}\xi_0$ for c_g:

$$\xi_g = \sum_{n \geq 0} \frac{\xi_{(\alpha_n, \alpha_{n+1})} - \xi_{(\alpha_{n+1}, \alpha_n)}}{\sqrt{\dim_q \alpha_n \dim_q \alpha_{n+1} \dim_q \alpha_1}}.$$

By property RD it lies in $M \otimes H_1$. $\implies \beta_1^{(2)}(\mathscr{A}_o(n)) = 0$ ■

The unitary case

The quasiclassical subgraph is a union of trees $\Rightarrow$ T injective on $\mathscr{K}_g'$.
Hence we have a unique path cocycle $c_g : \mathscr{A} \to \mathscr{K}_g'$.

Let $\gamma \in M_n \otimes \mathscr{A}$ be the fundamental corepresentation of $\mathscr{A}_u(n)$.
We consider $\alpha_k = \gamma^k$ and $\beta_k = \gamma\bar{\gamma}\gamma\cdots$.

Proposition

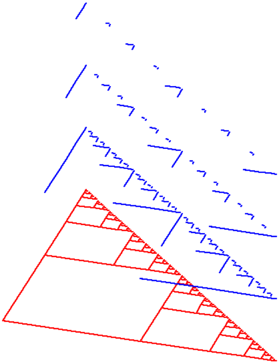
We have $\|(\mathrm{id}\otimes c_g)(\alpha_k)\| \geq C\sqrt{k}$ and $\|(\mathrm{id}\otimes c_g)(\beta_k)\| \leq D$ for all k and constants $C, D > 0$.

As a result c_g is neither trivial (bounded) nor proper.

$$\mathscr{A}_u(n) \text{ non-amenable} \implies \beta_1^{(2)}(\mathscr{A}_u(n)) \neq 0 \quad \blacksquare$$

The unitary case

Heuristically, the Proposition holds because there is no multiplicity above the zigzag path (β_k), and a lot of multiplicity above the straight line (α_k):



Work in progress

- Is $M = \lambda(\mathscr{A}_o(n))''$ strongly solid ?
- Is $M = \lambda(\mathscr{A}_o(n))''$ L^2-rigid ?