Planar algebras and random lattice Potts model

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work in collobaration with A. Guionnet, V. Jones, D. Shlyakhtenko.

Let $N \subset M$ be factors of type II_1. Jones in 1983 showed that the possible values of $[M : N] =$ are $\{4\cos^2 \pi/n, n \geq 3\} \cup [4, \infty)$.

If $E_N \in L^2(M, \mathrm{tr})$ is the orthogonal projector onto N, then $[M : N] = (\mathrm{tr}\, E_N)^{-1}$. Jones' basic construction:

$$M_0 = N \qquad M_1 = M \qquad M_{i+1} = \langle M_i, E_{M_{i-1}} \rangle \cong \underbrace{M \otimes_N M \otimes_N \cdots \otimes_N M}_{i+1}$$

Relates to the Temperley–Lieb algebra: the $e_i = \delta E_{M_{i-1}}$ satisfy

$$e_i^2 = \delta e_i \qquad e_i e_{i\pm 1} e_i = e_i \qquad e_i e_j = e_j e_i \quad |i-j| > 1$$

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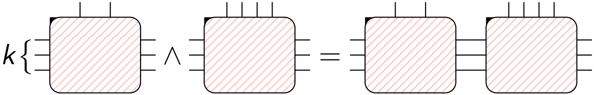
Lots more work in this direction (Haagerup, Popa, Ocneanu).

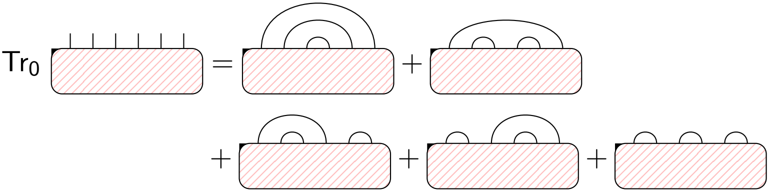
Importance of *higher relative commutants* $M_i' \cap M_j = \{x \in M_j : xy = yx \ \forall y \in M_i\}$ which extend the Temperley–Lieb construction. In Jones' language, $M_{0/1}' \cap M_j$ form a planar algebra (containing as subalgebra the Temperley–Lieb algebra). Encoded in a bipartite graph (the *principal graph*), δ being its Perron–Frobenius eigenvalue.

Inverse question: given $M_i \cap M_j'$, can we recover $N \subset M$? Answered affirmatively by Popa (1995).

Planar algebra: $P = \bigoplus_{k\geq 0} P_k$, where P_k is "blobs" with $2k$ legs (P_0 is scalars). A. Guionnet, V. Jones, D. Shlyakhtenko (2008) introduced a new product on $\bigoplus_{\ell\geq k} P_\ell$:



and a new trace: (note similarity with GUE)



They then show that the completion M_k of $\bigoplus_{\ell \geq k} P_\ell$ for this trace is a II_1 factor and the tower $M_0 \subset M_1 \subset \cdots$ has the desired properties.

We go further and study the analogue of non-Gaussian matrices: we paste arbitrary tangles *which respect bicoloration*:

$$\mathrm{Tr}_t(S) = \sum_{n_1,\ldots,n_k=0}^{\infty} \prod_{i=1}^{n_k} \frac{t_i^{n_i}}{n_i!} \sum_{P \in P(n_1,\ldots,n_k,S)} \delta^{\#\text{ loops in } P}$$

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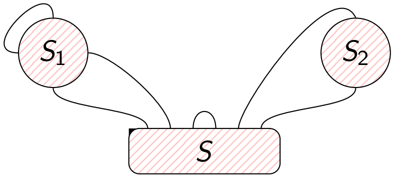

We show

Theorem (Guionnet, Jones, Shlyakhtenko, Z-J)

Let P be a finite-depth subfactor planar algebra and let $S_1, \ldots, S_k$ be elements of P. Then, for t small enough, Tr_t is a tracial state on P, as a limit of matrix models.

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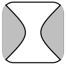
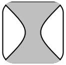
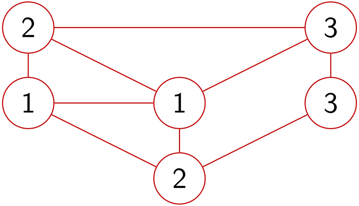
$S_1 =$  $S_2 =$ 

Let $\Gamma = (V, E)$ be an arbitrary graph and Q a positive integer.

$$\text{Configurations} = \text{maps } \sigma \text{ from } V \text{ to } \{1, \ldots, Q\}$$

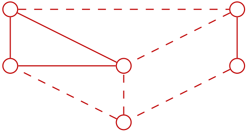
$$\text{Hamiltonian} = -K \sum_{\{i,j\} \in E} \delta_{\sigma_i, \sigma_j}$$



The partition function is

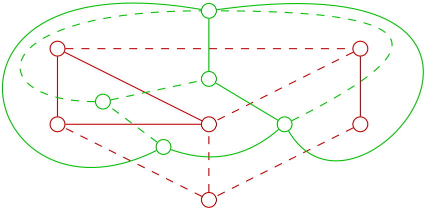
$$
\begin{aligned}
Z_\Gamma &= \sum_{\sigma:V\to\{1,\ldots,Q\}} \exp(K \sum_{\{i,j\}\in E} \delta_{\sigma_i,\sigma_j}) \\
&= \sum_{\sigma:V\to\{1,\ldots,Q\}} \prod_{\{i,j\}\in E} (1+v\delta_{\sigma_i,\sigma_j}) \qquad v := \exp(K)-1 \\
&= \sum_{E'\subset E} \sum_{\sigma:V\to\{1,\ldots,Q\}} \prod_{\{i,j\}\in E'} v\delta_{\sigma_i,\sigma_j} \\
&= \sum_{E'\subset E} v^{\#\text{ bonds}} Q^{\#\text{ clusters}}
\end{aligned}
$$

bonds=edges in E', clusters=connected components of the subgraph (V, E')



$\longrightarrow \quad v^4\, Q^3$

Assume Γ is embedded into the sphere ("planar map").
In particular, Γ is promoted to $\Gamma = (V, E, F)$.
There is a dual planar map $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \tilde{F})$, $\tilde{V} \cong F$, $\tilde{E} \cong E$, $\tilde{F} \cong V$.



Then we shall see that

$$Z_{\tilde{\Gamma}}(Q, v) \propto Z_{\Gamma}(Q, Q/v)$$

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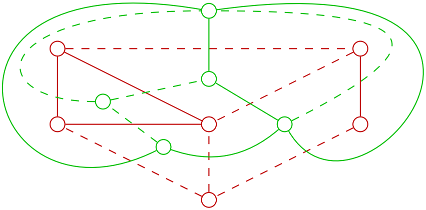

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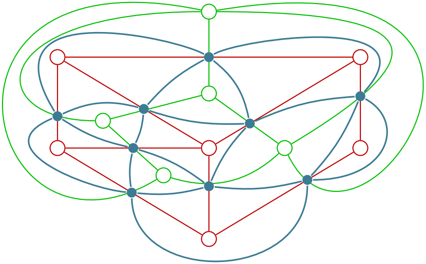
There is also a medial planar map $\Gamma_m = (V_m, E_m, F_m)$ with $V_m \cong E$, $F_m \cong V \sqcup F$:



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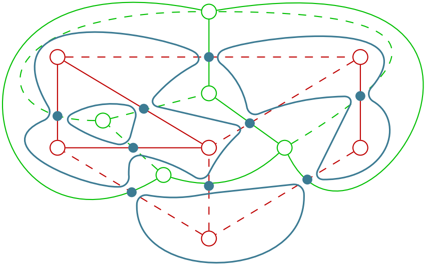

Splitting a vertex:

Each cluster is surrounded by $(2 + \#\text{ bonds} - \#\text{ vertices})$ loops. Therefore,

$$\#\text{ loops} = 2\#\text{ clusters} + \#\text{ bonds} - \#V$$

and finally

$$Z_\Gamma \propto \sum_{\substack{\text{loop}\\\text{configs}\\\text{on }\Gamma_m}} \sqrt{Q}^{\#\text{ loops}} \left(\frac{v}{\sqrt{Q}}\right)^{\#bonds}$$

(loop configuration=splitting of each vertex)

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We consider **dynamical** random lattices, that is

$$Z(x, y, Q, v) = \sum_{\Gamma=(V,E,F)} \frac{x^{\#E} y^{\#V}}{\text{symmetry factor}} \, Z_\Gamma(Q, v)$$

The summation is over arbitrary connected planar maps.

x and y are new parameters that control the typical size of the map; in what follows we only use x. (in the language of quantum gravity, it is the cosmological constant)

The duality $\Gamma \leftrightarrow \tilde{\Gamma}$ of the Potts model now becomes a symmetry of the model (at $v = \sqrt{Q}$)!

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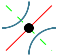
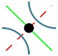
The equivalence to the loop model allows to state that

$$Z = \sum_{\Gamma_m} \frac{1}{\text{symmetry factor}} \sum_{\substack{\text{loop}\\ \text{configs}}} n^{\#\ \text{loops}} \alpha^{\#}$$  $$\beta^{\#}$$ 

where the summation is restricted to 4-valent planar maps, and

$$n = \sqrt{Q} \qquad \frac{\alpha}{\beta} = \frac{v}{\sqrt{Q}} \qquad \beta = x$$

Consider the following *formal* matrix integral:

$$I_N = \int \prod_{a=1}^{n} dM_a dM_a^\dagger \exp \Big[N \,\mathrm{tr} \Big(-\sum_{a=1}^{n} M_a M_a^\dagger + \frac{\alpha}{2} \sum_{a,b=1}^{n} M_a M_a^\dagger M_b M_b^\dagger + \frac{\beta}{2} \sum_{a,b=1}^{n} M_a^\dagger M_a M_b^\dagger M_b \Big) \Big]$$

over $N \times N$ complex matrices.

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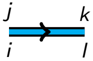
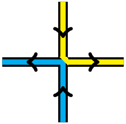
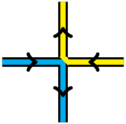
$$\left\langle (M_a)_{ij} (M_b)^\dagger_{kl} \right\rangle_0 = \delta_{ab}\delta_{il}\delta_{jk} =$$ 

$$\mathrm{tr}(M_a M_a^\dagger M_b M_b^\dagger) =$$ 

$$\mathrm{tr}(M_a^\dagger M_a M_b^\dagger M_b) =$$ 

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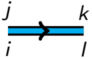

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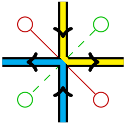

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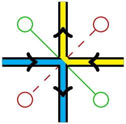

- The only use of the orientation of the edges is to distinguish Γ from $\tilde{\Gamma}$ in the original Potts language. For $\alpha \neq \beta$ this is important! For $\alpha = \beta$ one can remove the orientation and get back to the so-called $O(n)$ matrix model.

- If one tried to introduce *crossing* vertices, i.e. , then the corresponding terms $\mathrm{tr}(M_a M_b^\dagger M_a M_b^\dagger)$ would break the $U(n)$ symmetry (only the $O(n)$ symmetry would survive).

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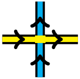
- If one tried to introduce *crossing* vertices, i.e. , then the corresponding terms $\mathrm{tr}(M_a M_b^\dagger M_a M_b^\dagger)$ would break the $U(n)$ symmetry (only the $O(n)$ symmetry would survive).

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&\qquad\qquad {}^{\left. + A\sum_{a=1}^n M_a M_a^\dagger + B\sum_{a=1}^n M_a^\dagger M_a\right)} \\
&= \int dA \int dB e^{N\,\mathrm{tr}\left(-\frac{1}{2\alpha}A^2 - \frac{1}{2\beta}B^2\right)} \det(1\otimes 1 - 1\otimes A - B\otimes 1)^{-n}
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NB: at this stage the Feynman diagram expansion can be done for arbitrary complex n.

Diagonalize the Hermitean matrices A and $B \to \{a_i\}, \{1 - b_i\}$

$$I_N = \int \prod_{i=1}^{N} da_i db_i \frac{\prod_{1\le i<j\le N}(a_j - a_i)^2 (b_j - b_i)^2}{\prod_{i,j=1}^{N}(a_i - b_j)^n} e^{N\sum_{i=1}^{N}(-\frac{1}{2\alpha}a_i^2 - \frac{1}{2\beta}(1-b_i)^2)}$$

Particles of two kinds, trapped in harmonic potentials, repelling particles of same kind and attracted ($n > 0$) to particles of different kind.

For sufficiently small α and β, the range of integration of the a_i and b_j can be restricted to intervals around 0 and 1 respectively, without changing the perturbative expansion, and such that the denominator never vanishes. The integral is then well-defined analytically.

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$$I_N = \int \prod_{i=1}^{N} da_i db_i \frac{\prod_{1 \le i < j \le N} (a_j - a_i)^2 (b_j - b_i)^2}{\prod_{i,j=1}^{N} (a_i - b_j)^n} e^{N \sum_{i=1}^{N} (-\frac{1}{2\alpha} a_i^2 - \frac{1}{2\beta} (1 - b_i)^2)}$$

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Define the resolvents of A and B:

$$G_A(a) = \lim_{N\to\infty} \frac{1}{N} \left\langle \mathrm{tr}\, \frac{1}{a-A} \right\rangle$$

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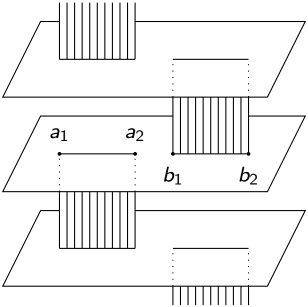
These functions satisfy the following saddle point equations:

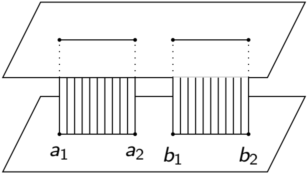
$$
\begin{aligned}
G_A(z+i0) + G_A(z-i0) &= P(z) + nG_B(z) \qquad && z \in [a_1, a_2] \\
G_B(z+i0) + G_B(z-i0) &= Q(z) + nG_A(z) \qquad && z \in [b_1, b_2]
\end{aligned}
$$

with $P(z) = z/\alpha$, $Q(z) = (1-z)/\beta$.

Analytically continuing these equations shows that G_A and G_B live on an infinite cover of the Riemann sphere:



Alternatively, they live on an infinite cover of the elliptic curve $y^2 = (z - a_1)(z - a_2)(z - b_1)(z - b_2)$:



We therefore introduce the parameterization

$$u(z) = \int_{b_2}^{z} \frac{dz}{\sqrt{(z-a_1)(z-a_2)(z-b_1)(z-b_2)}}$$

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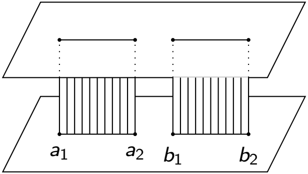

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More precisely, appropriate linear combinations of G_A and G_B:

$$G_\pm(u) = q^{\pm 1} G_A(u) - G_B(u) \pm \frac{1}{q - 1/q}(P(u) + q^{\pm 1} Q(u))$$

are sections of certain line bundles over this elliptic curve:

$$\begin{aligned} G_\pm(u + \omega_1) &= G_\pm(u) \\ G_\pm(u + \omega_2) &= q^{\pm 2} G_\pm(u) \end{aligned}$$

Here, $n = q + q^{-1}$, $|n| \neq 2$.

G_+ is meromorphic with only poles at $\pm u_\infty$, the two images of $z = \infty$. It can be expressed in terms of the theta function:

$$\Theta(u) = 2\sum_{k=0}^{\infty} e^{i\pi\frac{\omega_2}{\omega_1}(k+1/2)^2} \sin(2k+1)\frac{\pi u}{\omega_1}$$

Theorem

$$G_+(u) = c_+\frac{\Theta(u - u_\infty - \nu\omega_1)}{\Theta(u - u_\infty)} + c_-\frac{\Theta(u + u_\infty - \nu\omega_1)}{\Theta(u + u_\infty)}$$

where $q = \exp(i\pi\nu)$, *and*

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Then $G_\pm$ satisfy:

$$G_\pm(u + \omega_1) = G_\pm(u)$$
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i.e. they are elliptic functions with periods $\omega_1, p\omega_2$.

We conclude that $G_A(u)$ (resp. $G_B(u)$) and $z(u)$, being both elliptic with same periods, satisfy an algebraic equation:

$$P_A(G_A, z) = 0 \qquad P_B(G_B, z) = 0$$

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   - the singularity develops before the two types of particles meet:

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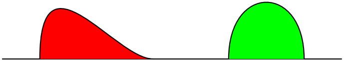

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     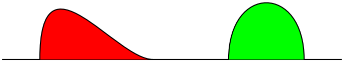 = pure gravity

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     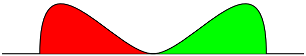 = critical statistical model

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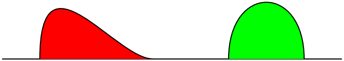
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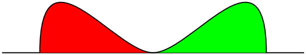
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