Genus expansion for some ensembles of random matrices

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GUE ensemble I

The problem

Let X be a random square $n \times n$ Gaussian Hermitian matrix. Entries x_{ij} are independent (up to transposition) complex-valued variables and

- $\mathbb{E}x_{ij} = 0$
- $\mathbb{E}x_{ij}x_{kl} = 1$, if $i = l$ and $j = k$
- $\mathbb{E}x_{ij}x_{kl} = 0$, otherwise.

What is the exact behavior of the spectral moments

$$m_k = \mathbb{E}\,\mathrm{tr}\,X^k?$$

Polygon

Trace

$$\mathbb{E}\,\mathrm{Tr}\,X^{2k} = \sum^{*} \mathbb{E} x_{i_0 i_1} x_{i_1 i_2} \cdots x_{i_{2k-1} i_0}, \tag{1}$$

where the sum $\sum^*$ is taken over all possible indices $i_0, i_1, \dots, i_{2k-1}$.

Wick formula

Let $(x_1, x_2, \dots, x_n)$ be a centered Gaussian vector and let $f_1, f_2, \dots, f_{2k}$ be linear functions of $x_1, x_2, \dots, x_n$. Then

$$\mathbb{E} f_1 f_2 \cdots f_{2k} = \sum \prod \mathbb{E}(f_i f_j),$$

where the sum is taken over all partitions of $\{1, 2, \dots, 2k\}$ into pairs.

Polygon

Polygon

Consider the product $x_{i_0 i_1} x_{i_1 i_2} \cdots x_{i_{2k-1} i_0}$ as a $2k$-gon. Then the partition into pairs corresponds to the pairwise gluing of all edges of this $2k$-gon.

The rule of gluing

The expectation $\mathbb{E} x_{ij} x_{kl}$ equals to 1 if $i = l$ and $j = k$ and it equals to 0 otherwise. So one has to glue the sides of a polygon in opposite directions. Such gluing gives an orientable surface.

Example $k = 2$

Example

$$\mathbb{E}\operatorname{Tr} X^4 = \sum^{*} \mathbb{E} x_{i_0 i_1} x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_0}.$$

- There are three partitions into pairs $\Rightarrow$ three gluings.
- Two of them give a **sphere**, the remaining gives a **torus**.
- Each gluing contributes in the $\sum^*$ as n^V, where V is the number of vertices of the obtained graph.
- Both of the planar graphs have 3 vertices, and the graph on the torus has 1 vertex.

Answer

$$\mathbb{E}\operatorname{Tr} X^4 = 2n^3 + n.$$

Example $k = 2$

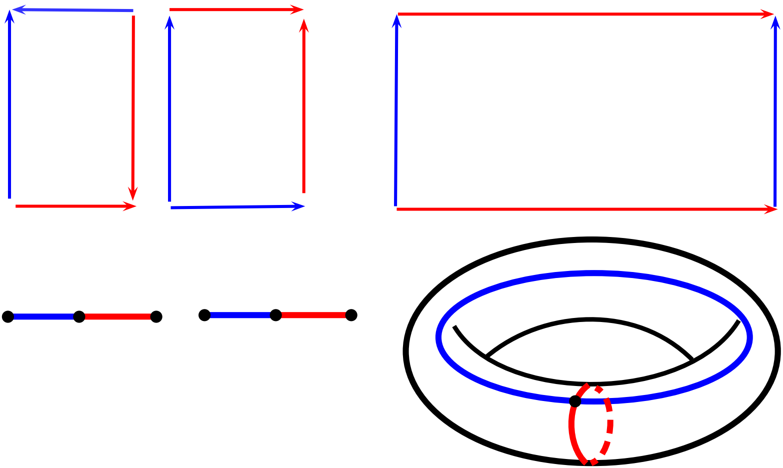

Figure: Two planar graphs and one graph on a torus

How many vertices does the glued graph have?

After gluing one obtains a surface and a graph embedded into the surface.
Euler formula

$$V - E + F = 2 - 2g,$$

where g is the genus of the surface.
The number of edges $E = k$, and there is only one face – the initial polygon. So,

$$V = k + 1 - 2g$$

Harer-Zagier Theorem

Harer-Zagier Theorem

$$\mathbb{E}\,\mathrm{Tr}\,X^{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} T(k,g) n^{k+1-2g},$$

where $T(k,g)$ is the number of ways to glue pairwise all the edges of a $2k$-gon to produce a surface of a given genus g.

Interpretation of Catalan numbers

Catalan numbers enumerate ways to glue a $2k$-gon into a sphere. So, it gives one more proof of the famous Wigner Semicircle Law.

The problem

The problem

Let X be a random square $n \times n$ Gaussian non-Hermitian matrix. Now all entries x_{ij} are independent random complex-valued variables and $\mathbb{E}x_{ij} = 0$, $\mathbb{E}x_{ij}\overline{x_{ij}} = 1$.
What is the limit behavior of the spectral moments $m_k = \mathbb{E}\,\mathrm{tr}(XX^*)^k$?

Polygon

In Wishart case the trace is $\mathbb{E}\,\mathrm{Tr}(XX^*)^k = \sum^* \mathbb{E}x_{i_0 i_1}x_{i_2 i_1}\cdots x_{i_0 i_{2k-1}}$. So a corresponding $2k$-gon is a directed graph.

Example

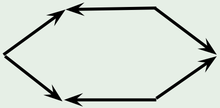

Figure: A hexagon, corresponding to $\mathbb{E}\,\mathrm{Tr}(XX^*)^3$

The rule of gluing

One has to glue the sides of a polygon, which have opposite directions.
There are k sides which are directed clockwise and k sides – counter-clockwise. So, the any gluing corresponds to some permutation.

Example

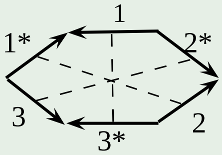

Figure: Permutation 312 corresponds to this gluing

Genus of Permutation

Definition

Let π be a permutation of $\{1, 2, \ldots, k\}$. Then the genus of the permutation π is

$$g(\pi) = 1/2(k + 1 - c(\pi) - c(\text{shift}(\pi^{-1})),$$

where shift stands for the composition with $234 \ldots k1$, π^{-1} stands for the inverse permutaion and $c(\pi)$ is the number of cycles in the permutation π.

Genus expansion for the Wishart case

Genus expansion

$$\mathbb{E}\,\mathrm{Tr}(XX^*)^k = \sum_{g=0}^{\lfloor k/2 \rfloor} P(k,g) n^{k+1-2g},$$

where $P(k,g)$ is the number of permutations of $\{1,2,\ldots,k\}$ of a given genus g.

Some Generalizations

The problem

Consider a random matrix $W_m = X_1 X_2 \cdots X_m (X_1 X_2 \cdots X_m)^*$, where $X_1, X_2, \ldots, X_m$ are independent Gaussian non-Hermitian matrices. What is the limit behaviour of the spectral distribution of W_m?

The first term

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\,\mathrm{Tr}\, W_m^k = FC(m,k) = \frac{1}{mk+1}\binom{mk+k}{k},$$

so-called Fuss-Catalan number.

Polygon

Let $m = 2$. In this case the trace is $\mathbb{E}\,\mathrm{Tr}(X_1 X_2 X_2{}^* X_1^*)^k = \sum^* \mathbb{E} x_{i_0 i_1} x_{i_1 i_2} \cdots x_{i_0 i_{2k-1}}$. So a corresponding $2k$-gon is a directed and multicolored graph.

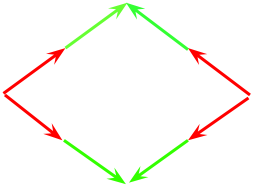

Figure: The case $k = 2$, $m = 2$

The rule of gluing

One has to glue the sides of a polygon, which have opposite directions and the same color.

Definition

Let (π_1, π_2) be a pair of permutations of $\{1, 2, \ldots, k\}$. Then the genus of the pair of permutations (π_1, π_2) $g : S_k \times S_k \mapsto \mathbb{N}$ is the genus of the corresponded surface.

Some numerical results

The case $m = 2$

$$\mathbb{E}\,\mathrm{tr}\,W_2 = 1$$

$$\mathbb{E}\,\mathrm{tr}\,W_2^2 = 3 + n^{-2}$$

$$\mathbb{E}\,\mathrm{tr}\,W_2^3 = 12 + 21n^{-2} + 3n^{-4}$$

$$\mathbb{E}\,\mathrm{tr}\,W_2^4 = 55 + 270n^{-2} + 231n^{-4} + 20n^{-6}$$

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Thanks

Thank you for your attention!