BROWNIAN MOTION ON MATRICES

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UNIFORM DISTRIBUTION ON AN ORBIT

Let $(\lambda_1, \ldots, \lambda_N) \in \mathbf{R}^N$, the orbit

$$\mathcal{O}_\lambda = \{UDU^* \mid U \in U(N)\}$$

of

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix}$$

by conjugation has a unique probability distribution invariant under $U(N)$.

HIZ FORMULA

The Fourier transform is given by Harish Chandra formula

$$\int_{U(N)} \exp(iTr(UDU^* A))dU = Z_N \frac{\text{dét}[(e^{i\lambda_j \mu_k})_{j,k}]}{V(\lambda)V(\mu)}$$

where μ_j are the eigenvalues of A and $V(\lambda)$ is the Vandermonde

$$V(\lambda) = \prod_{j<k} (\lambda_k - \lambda_j)$$

The Fourier transform is determined by its values on diagonal matrices A.

Remark: the formula is given by stationnary phase method.

DUISTERMAAT-HECKMAN MEASURE

For $A = diag(a_1, \ldots, a_N)$

$$\begin{aligned} F(a_1, \ldots, a_N) &= \int_{U(N)} \exp(iTr(UDU^* A))dU \\ &= \int_{U(N)} \exp(i \sum_j a_j (UDU^*)_{jj})dU \end{aligned}$$

is the Fourier transform of the distribution of

$$((UDU^*)_{11}, \ldots, (UDU^*)_{NN})$$

This measure on $\mathbf{R}^N$ is supported by the hyperplane

$$\sum_i x_i = Tr(D)$$

It is the Duistermaat-Heckman measure.

SOME PROPERTIES OF DH MEASURE

The support of the Duistermaat-Heckman measure is the convex hull of the points $(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(N)})$, where $\sigma \in S_N$.

It has a piecewise polynomial density on this set.

It is the image by an affine map of Lebesgue measure on a convex polytope of dimension $\frac{N(N-1)}{2}$.

This property can be established by symplectic geometry techniques (toric degeneration of Schubert varieties, convexity properties of moment mappings).

EXAMPLE; N=2

$D = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ $\qquad \lambda > 0$, the orbit is

$$\begin{pmatrix} x & z \\ \bar{z} & -x \end{pmatrix}$$

such that $x^2 + |z|^2 = \lambda^2$. A sphere S^2 of radius λ.
D-H measure is the projection of uniform measure on S^2 onto a diameter.
It is Lebesgue measure on $[-\lambda, \lambda]$, for $A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$

$$\int_{U(2)} \exp(\frac{i}{2} Tr(UDU^* A) dU = Z_2 \frac{\det \begin{pmatrix} e^{\frac{i}{2}\lambda a} & e^{-\frac{i}{2}\lambda a} \\ e^{-\frac{i}{2}\lambda a} & e^{\frac{i}{2}\lambda a} \end{pmatrix}}{i\lambda a}$$

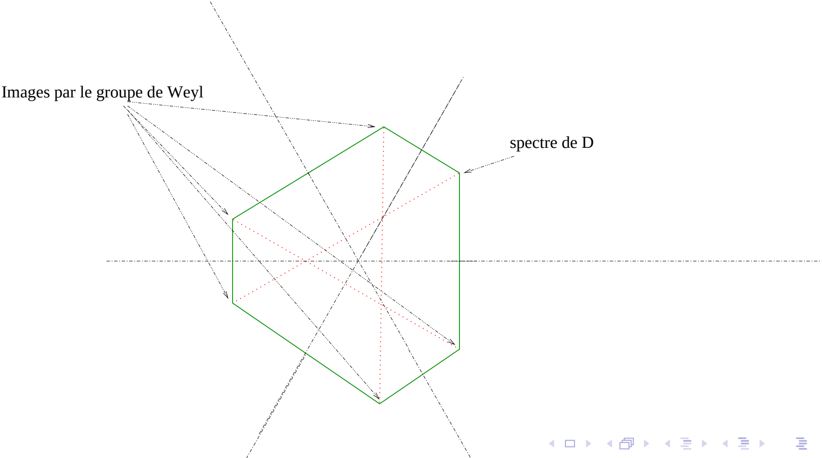
$$= \frac{\sin(\lambda a)}{\lambda a} = \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} e^{iax} dx$$

N=3

Take

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

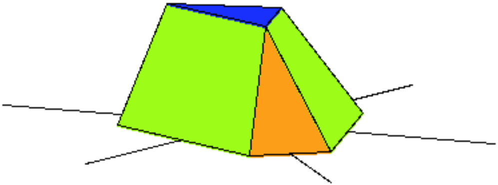
the measure is supported by a convex set.



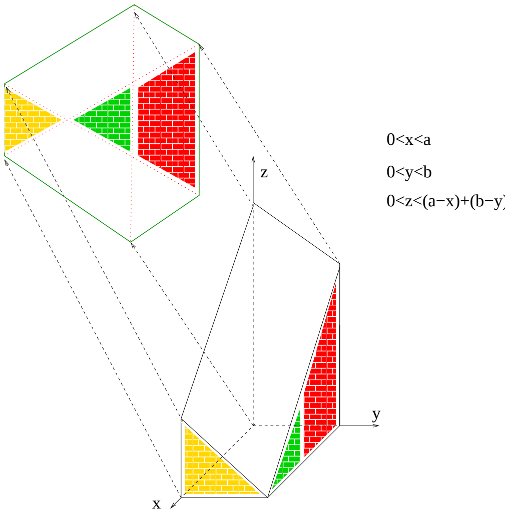
Density is piecewise polynomial with degree

$$\frac{(N-1)(N-2)}{2}$$

degree= 1 for N=3



This measure is the image by an affine map affine of Lebesgue measure on a convex polytope



BROWNIAN INTERPRETATION

OF DUISTERMAAT-HECKMAN MEASURE

$B = (B_{ij}(t))_{1\leq i,j\leq N}$ $U(N)$-invariant brownian motion on $N \times N$ hermitian matrices

Duistermaat-Heckmann measure is the conditional distribution of $(B_{11}(t), \ldots, B_{NN}(t))$ knowing that $B(t)$ has spectrum $(\lambda_1, \ldots, \lambda_N)$.

Let $B = (B_{ij}(t))_{1\leq i,j\leq N}$ Brownian motion in hermitian matrices
The eigenvalues $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_N(t)$ follow a diffusion process in the cone $C = \{x_1 \geq x_2 \geq \ldots \geq x_N\}$.
The Laplace operator on the cone with Dirichlet boundary conditions on C has a unique positive harmonic function

$$h(x) = \prod_{j<k}(x_k - x_j)$$

Brownian motion conditionned to stay in the cone has infinitesimal generator

$$\frac{1}{2h}\Delta h$$

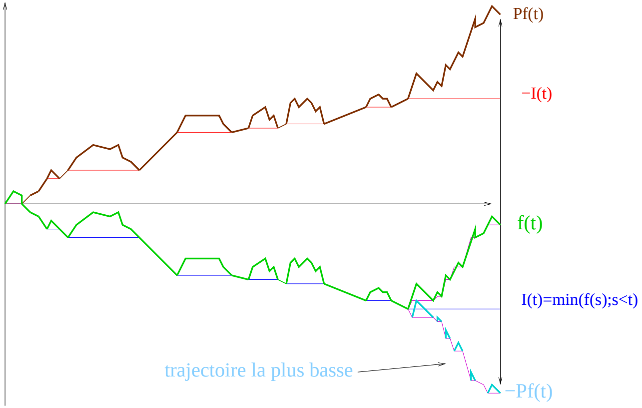
and semigroup

$$q_t(x, y) = \frac{h(y)}{h(x)} \sum_{\sigma\in S_N} \varepsilon(\sigma) \frac{e^{-\frac{|x-\sigma(y)|^2}{2t}}}{(2\pi t)^{N/2}}$$

PITMAN TRANSFORM

$f : [0, t] \to \mathbf{R}$ continuous,

$f(0) = 0, \qquad Pf(s) = f(s) - 2\inf_{0 \le u \le s} f(u)$



PITMAN THEOREM

$(B_t)_{t\geq 0}$ = brownian motion, then

$$PB(t) = B_t - 2 \inf_{0\leq s\leq t} B_s$$

is a Bessel3 process= the norm of a three dimensional Brownian motion.

GENERALIZED PITMAN TRANSFORM

V = euclidian space, $a \in V$, $\langle a, a \rangle = 1$. Pitman transformation in the direction a is:

$$P_a f(t) = f(t) - 2 \inf_{0 \leq s \leq t} \langle f(s), a \rangle a$$

$P_a P_a f = P_a f$.

BRAID RELATIONS

Let $a, b \in V$, of norm 1, $\langle a, b\rangle = -\cos\theta$. If $n\theta \leq \pi$ then

$$(n \text{ terms}) \quad P_a P_b P_a \dots f(t) = f(t) -$$

$$2 \inf_{t \geq s_1 \geq \ldots \geq s_n \geq 0} [\frac{\sin\theta}{\sin\theta}\langle f(s_1), a\rangle + \frac{\sin 2\theta}{\sin\theta}\langle f(s_2), b\rangle + \frac{\sin 3\theta}{\sin\theta}\langle f(s_3), a\rangle + \ldots$$

$$-2 \inf_{t \geq s_1 \geq \ldots \geq s_{n-1} \geq 0} [\frac{\sin\theta}{\sin\theta}\langle f(s_1), b\rangle + \frac{\sin 2\theta}{\sin\theta}\langle f(s_2), a\rangle + \frac{\sin 3\theta}{\sin\theta}\langle f(s_3), b\rangle +$$

If $\theta = \pi/n$ then

$$P_a P_b P_a \ldots = P_b P_a P_b \ldots \qquad (n \text{ termes})$$

(G, S) Coxeter group.
G is generated by hyperplane reflexion $s_i \in S$, such that

$$s_i s_j s_i \ldots = s_j s_i s_j \ldots$$

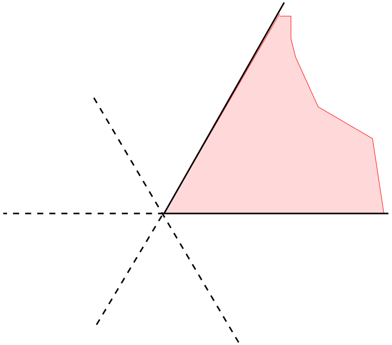
with n_{ij} factors in each term
Chose a fundamental domain for G=a convex cone intersection of half-spaces $C = \cap_{i \in S} \{\langle a_i, x \rangle \geq 0\}$ where a_i is orthogonal to H_i, the hyperplane of s_i.

THE SYMMETRIC GROUP

S_n acts on $\mathbf{R}^n$ by permutation of coordinates.
$s_i : x_i \to x_{i+1}$ satisfy

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \quad s_i s_j = s_j s_i, \; |i-j| > 1$$

$(1,1,1,\ldots,1)$ is invariant as well as its orthogonal $\sum_i x_i = 0$.
$x_1 > x_2 > \ldots > x_n$ is a fundamental domain.
Example: $n = 3$



GENERALIZED PITMAN OPERATORS

Because of the braid relations, for every $w \in G$ one can define a Pitman operator. If

$$w = s_{i_1} \dots s_{i_k}$$

is a reduced decomposition then

$$P_w = P_{s_{i_1}} P_{s_{i_2}} \dots P_{s_{i_k}}$$

depends only on w.

Let $w_0 \in G$ be the longest element.
EXAMPLE: if $G = S_N$ then $w_0(i) = N + 1 - i$.
For $N = 3$ one has $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$.

Proposition

For all f, $P_{w_0} f$ takes values in the fundamental domain, the cone C.

GENERALIZED PITMAN THEOREM

Theorem *1. If $B(t)$ is a Brownian motion in V then $P_{w_0}B$ is a Brownian motion conditioned to stay in C.*

2. The conditional distribution of $B(t)$ knowing $P_{w_0}B(s); s \leq t$ is the Duistermaat-Heckman measure on the convex hull of $g(P_{w_0}B(t)); g \in G$.

Let $w_0 = s_{i_1} \ldots s_{i_n}$ a reduced decomposition
$w_k = s_{i_{k+1}} \ldots s_{i_n}$ then

$$P_{w_{k-1}} B(t) = P_{w_k} B(t) - 2 \inf \langle a_{i_{k+1}}, P_{w_k} B(s) \rangle a_{i_{k+1}}$$

Let $x_k = -2 \inf \langle a_{i_{k+1}}, PB_{w_k}(s) \rangle$
then

$$P_{w_0} B(t) = B(t) + \sum_k x_k a_{i_{k+1}}$$

Theorem The conditional distribution of $(x_1, \ldots, x_n)$ knowing $(P_{w_0} B(s); s \leq t)$, is Lebesgue on a certain convex polytope.
The Duistermaat-Heckman measure is the image by

$$(x_1, \ldots, x_n) \mapsto P_{w_0} B(t) - \sum_k x_k a_{i_{k+1}}$$