Multiplication law and S-transform for non-hermitian random matrices

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Outline

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   - Multiplication law
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   - Addition law
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4. Summary

Probability: Addition law

- Knowing independent $p_a(x_a)$ and $p_b(x_b)$, we want to infer $p(s)$, where $s = x_a + x_b$
- $p(s) = \int dx_a dx_b p_a(x_a) p_b(x_b) \delta(s - (x_a + x_b)) = \int dx p_a(x) p_b(s - x)$
- Fourier transform $f(k) = \int dk e^{ikx} p(x) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \int p(x) x^n dk = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} m_n$ unravels the convolution, $f(k) = f_a(k) \cdot f_b(k)$
- $r(k) \equiv \ln f(k) = \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} c_n$ generates cumulants
- **Addition law** $r(k) = r_a(k) + r_b(k)$, i.e. $c_n = c_n^{(a)} + c_n^{(b)}$
- Ex.: Gaussian pdf $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \equiv N(0,1)$ (only $c_2 \neq 0$) $N_a(0,1) \oplus N_b(0,1) = N(0, \sqrt{2})$

Probability: Multiplication law

- Knowing independent $p_a(x_a)$ and $p_b(x_b)$, we want to infer $p(r)$, where $r = x_a \cdot x_b$
- $p(r) = \int dx_a dx_b p_a(x_a) p_b(x_b) \delta(r - x_a \cdot x_b) = \int \frac{dx}{x} p_a(\frac{r}{x}) p_b(x)$
- Mellin transform $m(t) = \int_0^\infty \frac{dx}{x} x^t p(x)$ factorizes above integral
- **Multiplication law** $m(t) = m_a(t) \cdot m_b(t)$
- Technical modification for negative random variables $m(t) = m_{++}(t) + m_{+-}(t) + m_{-+}(t) + m_{--}(t)$ [Epstein;1948]
- Inverse Mellin transform $p(r) = \int_\Gamma m(t) t^{-r} dt$ yields the result
- Ex.: $N_a(0,1) \otimes N_b(0,1) = \frac{1}{\pi} K_0(x)$ (MacDonald function), $\lim_{|x| \to 0} K_0(x) \sim -\ln|x|$, $\lim_{|x| \to \infty} K_0(x) \sim \sqrt{\frac{\pi}{2|x|}} e^{-|x|}$

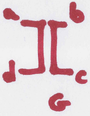
RMT: Matrix-valued probability calculus (real eigenvalues)

- $d\mu(X) \equiv P(X)dX = e^{-N\mathrm{tr}V(X)}dX$
- Key question: statistics of the spectra of X, e.g. $\rho(\lambda) = \frac{1}{N}\langle \mathrm{tr}\delta(\lambda - X)\rangle$
- $d\mu(\{\lambda_i\}) = C_N^{\beta} e^{-N\sum_j V(\lambda_j)} \prod_{i<j} (\lambda_i - \lambda_j)^{\beta} \prod_j d\lambda_j$, eigenvalues *interact* with each other ($\beta = 1(2)$ for real(complex) matrices).
- Large N simplifications: theoretical (planar "graphs" dominate in the guise of 't Hooft expansion), practical ($8 \sim \infty$)
- Complex Green's function generates spectral moments M_k: $G(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z\mathbf{1}_N - X}\right\rangle = \sum_{k=0} \frac{M_k}{z^{k+1}}$ where $M_k = \frac{1}{N}\left\langle \mathrm{tr}X^k\right\rangle = \int \rho(\lambda)\lambda^k d\lambda$
- Analytic properties are crucial: $\rho(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0} \Im G(z)|_{z=\lambda+i\epsilon}$

RMT: Matrix-valued probability calculus (real eigenvalues)

- What about spectral cumulants C_k?
- R-transform [Voiculescu;1986] $R(z) = \sum_{k=1} C_k z^{k-1}$
- $G(R(z) + \frac{1}{z}) = z$ and $R(G) + \frac{1}{G} = z$
- Physicist "proof": RMT is a QFT in $0+0$ dimensions, hence $Z = \int dX e^{-N \mathrm{tr} V(X)}$
  1. 't Hooft double line notation for Wick expansion: Propagator $< XX > \sim 1/N$, each vertex brings N, each loop brings N
  2. Only planar "Feynman graphs" survive $N \to \infty$ limit
  3. We define 1PI "self energy" $\Sigma(z)$ as $G(z) = \frac{1}{z - \Sigma(z)}$
  4. Self-energy gets contributions from renormalized propagators $1/z \to G(z)$ and renormalized vertices C_k, so $\Sigma(z) = \sum_{k=1} C_k G(z)^{k-1} = R(G(z))$ (Schwinger-Dyson equation)

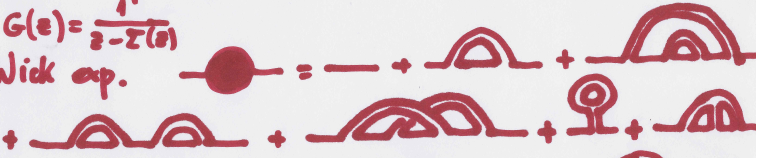
DIAGRAMMAR

- Feynman rules:  $\frac{1}{z}\delta^a_b$ 

$\langle X^a_b X^c_d \rangle = \frac{1}{N}\delta^a_d \delta^c_d$

- Self-energy (1PI) 

$G(z) = \frac{1}{z-\Sigma(z)}$

- Wick exp. 

- S-D eq. 

$$\Sigma(z) = c_1 + c_2 G(z) + c_3 G^2(z) + \ldots = \sum_{k=1} c_k G^{k-1} = R(G)$$

R-transform: Addition law

- Knowing $\rho_A(\lambda)$ and $\rho_B(\lambda)$ for independent (free) RM ensembles, we want to infer $\rho_{A+B}(\lambda)$, i.e. calculate $G_{A+B}(z) = \frac{1}{N}\int dX_A dX_B P_A(X_A) P_B(X_B) \mathrm{tr} \frac{1}{z\mathbf{1}_N - (X_A + X_B)}$
- Non-commutative convolution, since $[X_A, X_B] \neq 0$
- **Addition law**: First, from the definition $G(R(z) + \frac{1}{z}) = z$ we read the corresponding transforms $R_A(z)$ and $R_B(z)$. Second we apply addition law $R_{A+B}(z) = R_A(z) + R_B(z)$. Third, we invert functionally $R_{A+B}(z)$ to get the desired result.
- For matrix analogue of the Gaussian distribution only $C_2 \neq 0$, i.e. $R_W(z) = C_2 z$. For $C_2 = 1/4$ Green's function reads $G_W(z) = 2(z - \sqrt{z^2 - 1})$, so $\rho_W(\lambda) = \frac{2}{\pi}\sqrt{1 - \lambda^2} \equiv W(0,1)$. Wigner semicircle $\leftrightarrow$ Gaussian.
- Ex.: $W_A(0,1) \bigoplus W_B(0,1) = W(0,\sqrt{2})$, in analogy to the classical case.

S-transform: Multiplication law

- Knowing $\rho_A(\lambda)$ and $\rho_B(\lambda)$ for independent (free) RM ensembles, we want to infer $\rho_{A\cdot B}(\lambda)$, i.e. calculate $G_{A\cdot B}(z) = \frac{1}{N}\int dX_A dX_B P_A(X_A) P_B(X_B) \mathrm{tr} \frac{1}{z\mathbf{1}_N - (X_A \cdot X_B)}$
- In general, for hermitian matrices, $(H_1 \cdot H_2)^\dagger \neq H_1 \cdot H_2$, so spectra are complex.
- For unitary random matrices, $(U_1 \cdot U_2)^\dagger \cdot U_1 \cdot U_2 = \mathbf{1}$, spectra on the unit circle $\lambda = e^{i\theta}$, analytic methods applicable.
- S-transform [Voiculescu;1987] $S(z)G(\frac{1+z}{z}S(z)) = z$
- **Multiplication law**: $S_{A\cdot B}(z) = S_A(z) \cdot S_B(z)$.
- S-transforms and R-transforms are related, alike Fourier and Mellin transforms are related.
- Alternative version for multiplication law [Janik;1997] $R_{A\cdot B}(g) = R_A(g_B) \cdot R_B(g_A)$ where $g_A = gR_A(g_B)$ and $g_B = R_B(g_A)g$

S-transform - preliminary: relation to R-transform

- $S(z) = \frac{1+z}{z}\chi(z)$, where $\chi(zG(z)-1) = \frac{1}{z}$.
- If $z \equiv yG(y)-1$, then $S(yG(y)-1) = \frac{1}{y-\frac{1}{G(y)}}$. Since $G(y) = \frac{1}{z-\Sigma(y)}$, we get $S(G(y)\Sigma(y)) = \frac{1}{\Sigma(y)}$. Since $\Sigma(y) = R(G(y))$, we arrive (after taking reciprocals of both sides) at $\frac{1}{S(G(y)R(G(y))} = R(G(y))$. Finally, changing variables again $z = G(y)$ we arrive at $R(z) = \frac{1}{S(zR(z))}$.
- Note that S transform can be defined only if $R(0) \neq 0$, (non-vanishing first moment)
- Last equation can be inverted: Let us define $y = zR(z)$. Then $S(y) = \frac{1}{R(\frac{y}{R(z)})} = \frac{1}{R(\frac{y}{R(\frac{y}{R(\ldots)})})}$, where z is recursively eliminated ad infinitum $S(y) = \frac{1}{R(yS(y))}$.
- Mutually inverse maps $z = yS(y)$ and $y = zR(z)$

S-transform - diagrammatics

- We consider $2N \times 2N$ block matrices
  $\mathcal{H} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \qquad \mathcal{H}^{2k} = \begin{pmatrix} (AB)^{2k} & 0 \\ 0 & (BA)^{2k} \end{pmatrix}$
- We define $\mathcal{G}(w) = \begin{pmatrix} \mathcal{G}_{11}(w) & \mathcal{G}_{12}(w) \\ \mathcal{G}_{21}(w) & \mathcal{G}_{22}(w) \end{pmatrix} = \frac{1}{N}\left\langle \mathrm{tr}_{\,\mathrm{b2}} \frac{1}{w\mathbf{1} - \mathcal{H}} \right\rangle$
- Block-trace definition $\mathrm{tr}_{\,\mathrm{b2}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{tr}\, A & \mathrm{tr}\, B \\ \mathrm{tr}\, C & \mathrm{tr}\, D \end{pmatrix}$
- Note that $G_{AB}(z = w^2) = \frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{z\mathbf{1} - AB} \right\rangle = \frac{\mathcal{G}_{11}(w)}{w}$
- Similarly, we define $\Sigma(w) = \begin{pmatrix} \Sigma_{11}(w) & \Sigma_{12}(w) \\ \Sigma_{21}(w) & \Sigma_{22}(w) \end{pmatrix}$, where
  $\mathcal{G}(w) = (w\mathbf{1}_2 - \Sigma(w))^{-1}$

S-transform - alternative formulation

- From the flow of indices ($\mathcal{H}_{12} \leftrightarrow A$, $\mathcal{H}_{21} \leftrightarrow B$) we get $\Sigma(w) = \begin{pmatrix} 0 & R_A(\mathcal{G}_{21}(w)) \\ R_B(\mathcal{G}_{22}(w)) & 0 \end{pmatrix}$ where $\mathcal{G}(w) = (w\mathbf{1}_2 - \Sigma(w))^{-1}$
- $\begin{pmatrix} \mathcal{G}_{11}(w) & \mathcal{G}_{12}(w) \\ \mathcal{G}_{21}(w) & \mathcal{G}_{22}(w) \end{pmatrix} = \mathcal{G}(w) = (w\mathbf{1}_2 - \Sigma(w))^{-1} = \begin{pmatrix} w & -R_A(\mathcal{G}_{21}(w)) \\ -R_B(\mathcal{G}_{22}(w)) & w \end{pmatrix}^{-1}$
- Inverting matrix we get ($w^2 = z$) $G_{AB}(z) = \frac{1}{z - \Sigma_{AB}(z)} = \frac{\mathcal{G}_{11}(w)}{w} = \frac{1}{z - R_A(\mathcal{G}_{21}(w))R_B(\mathcal{G}_{12}(w))}$ $\mathcal{G}_{12} = G_{AB}R_A(\mathcal{G}_{21})$, $\mathcal{G}_{21} = G_{AB}R_B(\mathcal{G}_{12})$
- Using $G_{AB}(z) = \frac{1}{z - R_{AB}(G_{AB})}$ we get the multiplication law.

Relation to "canonical" form of S

- $G_{AB} \equiv g$, $\mathcal{G}_{12} \equiv g_A$, $\mathcal{G}_{21} \equiv g_B$
- Algorithm: Three equations with three *complex* variables: $R_{A \cdot B}(g) = R_A(g_B) \cdot R_B(g_A)$ where $g_A = gR_A(g_B)$ and $g_B = R_B(g_A)g$
- To unravel equations, we define $y = gR_{AB}(g)$. Then $g_B = gR_B(g_A) = g\frac{R_{AB}(g)}{R_A(g_B)} = \frac{y}{R_A(g_B)} = \frac{y}{R_A(\frac{y}{R_A(\dots)}))}$
- $$R_{AB}\left(\frac{y}{R_{AB}\left(\frac{y}{R_{AB}(\dots)}\right)}\right) = R_A\left(\frac{y}{R_A\left(\frac{y}{R_A(\dots)}\right)}\right) R_B\left(\frac{y}{R_B\left(\frac{y}{R_B(\dots)}\right)}\right)$$
- But one has to assume $R_i(0) \neq 0$ for $i = A, B, AB$
- Taking reciprocal of the above equation we arrive at $S_{AB}(y) = S_A(y) \cdot S_B(y)$

Non-hermitian case - electrostatic analogy (Dyson gas)

Analytic methods break down, since spectra are complex $\rho(z,\bar{z}) = \frac{1}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i)\right\rangle$.

- Potential $\Phi(z,\bar{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\,\ln[(z\mathbf{1}_N - X)(\bar{z}\mathbf{1}_N - X^\dagger)+\epsilon^2\mathbf{1}_N]\right\rangle$
- Poisson law $\frac{\partial^2\Phi}{\partial z\partial\bar{z}} = \pi\rho(z,\bar{z})$, since $\delta^{(2)}(z) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon^2}{(|z|^2+\epsilon^2)^2}$
- Electric field $G(z,\bar{z}) = \frac{\partial\Phi}{\partial z}$
- Gauss law $\frac{1}{\pi}\partial_{\bar{z}}G(z,\bar{z}) = \rho(z,\bar{z})$ [Brown;1986],[Sommers,Crisanti,Sompolinsky,Stein;1988]
- Bad news: $G(z,\bar{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle \frac{1}{N}\mathrm{tr}\,\frac{\bar{z}-X^\dagger}{(z\mathbf{1}_N-X)(\bar{z}\mathbf{1}_N-X^\dagger)+\epsilon^2}\right\rangle$
- No similarity to the hermitian case $G(z) = \left\langle \frac{1}{N}\mathrm{tr}\,\frac{1}{z\mathbf{1}_N - X}\right\rangle$
- Important in applications (dissipation, directed percolation, lagged correlations..), interesting in mathematics [Biane,Lehner;1999]

Non-hermitian case - Remedy: $\mathrm{tr} \ln A = \ln \det A$

$$\mathrm{tr} \ln[(z\mathbf{1}_N - X)(\bar{z}\mathbf{1}_N - X^\dagger) + \epsilon^2 \mathbf{1}_N] = \ln \det \begin{pmatrix} z\mathbf{1}_N - X & i\epsilon\mathbf{1}_N \\ i\epsilon\mathbf{1}_N & \bar{z}\mathbf{1}_N - X^\dagger \end{pmatrix}$$

- Duplication trick
  [Janik,MAN,Papp,Zahed;1996],[Feinberg,Zee;1997]
  $\mathrm{tr}_{\,\mathrm{b}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \mathrm{tr}\, A & \mathrm{tr}\, B \\ \mathrm{tr}\, C & \mathrm{tr}\, D \end{pmatrix}$
- $\mathcal{Z}_N = \begin{pmatrix} z & i\epsilon \\ i\epsilon & \bar{z} \end{pmatrix} \otimes \mathbf{1}_N \equiv \mathcal{Z} \otimes \mathbf{1}_N \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$
- 2 by 2 objects $\mathcal{G}(\mathcal{Z}) = \frac{1}{N} \left\langle \mathrm{tr}_{\,\mathrm{b}} \frac{1}{\mathcal{Z}_N - \mathcal{X}} \right\rangle = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}} \end{pmatrix}$

Benefits of the duplication trick

- Self-energy Σ is a 2 by 2 matrix $\mathcal{G}(\mathcal{Z}) = \frac{1}{\mathcal{Z}-\Sigma(\mathcal{Z})}$
- Analogue of R-transform exists $\mathcal{R}(\mathcal{G}) + \frac{1}{\mathcal{G}} = \mathcal{Z}$
- Addition law holds $\mathcal{R}_{A+B}(\mathcal{Z}) = \mathcal{R}_A(\mathcal{Z}) + \mathcal{R}_B(\mathcal{Z})$
- Upper-left corner of $\mathcal{G}$, i.e. $\mathcal{G}_{11} = G(z, \bar{z})$, so $\frac{1}{\pi}\partial_{\bar{z}}\mathcal{G}_{11} = \rho(\lambda)$
- Condition $\mathcal{G}_{1\bar{1}}\mathcal{G}_{\bar{1}1} = 0$ provides the equation for the boundary of eigenvalues
- Ex.: $W_a(0,1) \bigoplus i \cdot W_b(0,1) =$ Ginibre-Girko ensemble (uniform distribution bounded by the circle $|z| = \sqrt{2}$)

Hidden algebraic structure unveiled

- Each generic 2 by 2 matrix Q which has appeared before has the structure $Q = \begin{pmatrix} z & i\bar{w} \\ iw & \bar{z} \end{pmatrix}$
- Q is a quaternion
- $Q = q_0 \mathbf{1}_2 + i\sigma_i q_i = \begin{pmatrix} q_0 + iq_3 & i(q_1 - iq_2) \\ i(q_1 + iq_2) & q_0 - iq_3 \end{pmatrix}$
- One can exploit the whole space of Q, instead of staying infinitesimaly close (ϵ) in transverse directions $(1, 2)$
- Algorithm how to embed Greens functions and R-transforms in quaternion space for any hermitian H, any $H_1 + iH_2$ [Jarosz,MAN;2004], any unitary U [Jarosz,Goerlich;2005], and several other cases.
- Examples: $\mathcal{R}_{GUE}(Q) = Q$, $\mathcal{R}_{G-G} \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & i\bar{b} \\ ib & 0 \end{pmatrix}$

Correspondence of the addition laws

Hermitian case

- real spectra
- Green's function $G(z)$ is complex
- $G(R(z) + \frac{1}{z}) = z$
- Addition law $R_{A+B}(z) = R_A(z) + R_B(z)$
- Analytic functions (cuts and poles as singularities)

Non-hermitian case

- complex spectra
- Green's function $\mathcal{G}(Q)$ is a quaternion
- $\mathcal{G}(\mathcal{R}(Q) + \frac{1}{Q}) = Q$
- Addition law $\mathcal{R}_{A+B}(Q) = \mathcal{R}_A(Q) + \mathcal{R}_B(Q)$
- Matrix-valued non-analytic functions

Does nonhermitian S transform exist?

Despite doubts if such construction is possible at all, several recent results on products of random matrices were suggesting the possibility that such law may exist, e.g.:

- Nonhermitian diffusion [Janik,Jurkiewicz,Gudowska-Nowak,MAN;2003],[Lohmayer,Neuberger,Wettig;2008], [Warchoł;2010]
- Products of centered complex matrices [Girko,Vladimirova;2009], [Burda, Janik, Wacław;2010],[Burda,Jarosz,Livan,MAN,Święch;2010]
- Time-lagged correlations [Biely,Thurner;2007],[Jarosz;2010]
- Additive laws for unitary ensembles [Goerlich,Jarosz;2004]
- Multiplication for vanishing mean ensembles [Rao,Speicher;2007]

Multiplication law for non-hermitian random matrices

- Multiplication law reads [Burda,Janik,MAN;2011a] :
- $\mathcal{R}_{A\cdot B}(\mathcal{G}_{A\cdot B}) = \mathcal{R}^L_A(\mathcal{G}_B)\cdot\mathcal{R}^R_B(\mathcal{G}_A)$ where $\mathcal{G}_A = \left[\mathcal{G}_{A\cdot B}\mathcal{R}^L_A(\mathcal{G}_B)\right]^L$ and $\mathcal{G}_B = \left[\mathcal{R}^R_B(\mathcal{G}_A)\mathcal{G}_{A\cdot B}\right]^R$
  where for generic Q we have $Q^L = UQU^\dagger$ ($Q^R = U^\dagger QU$) and $U = e^{i\frac{\phi}{2}\frac{\sigma_3}{2}}$ with $\phi = \mathrm{Arg}\, z$.
- Note that order matters, since matrices (quaternions) do not commute.
- Three (matrix-valued) equations for three *quaternion* variables.
- In the case when $[\mathcal{G},\mathcal{R}] = 0$, addition law gets reduced to S-transform by Voiculescu, i.e to the mutiplication law for complex functions $R_{A\cdot B}(G_{A\cdot B}) = R_A(G_B)\cdot R_B(G_A)$ where $G_A = G_{A\cdot B}R_A(G_B)$ and $G_B = R_B(G_A)G_{A\cdot B}$

Elements of the construction (I)

- We have product of random matrices and spectrum is complex, so we combine *both* duplication tricks:
- $\mathcal{H} = \begin{pmatrix} 0 & A & 0 & 0 \\ B & 0 & 0 & 0 \\ 0 & 0 & 0 & B^\dagger \\ 0 & 0 & A^\dagger & 0 \end{pmatrix}_{4N \times 4N}$
- $\mathcal{G}(\mathcal{W}) \equiv \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{G}_{1\bar{1}} & \mathcal{G}_{1\bar{2}} \\ \mathcal{G}_{21} & \mathcal{G}_{22} & \mathcal{G}_{2\bar{1}} & \mathcal{G}_{2\bar{2}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}2} & \mathcal{G}_{\bar{1}\bar{1}} & \mathcal{G}_{\bar{1}\bar{2}} \\ \mathcal{G}_{\bar{2}1} & \mathcal{G}_{\bar{2}2} & \mathcal{G}_{\bar{2}\bar{1}} & \mathcal{G}_{\bar{2}\bar{2}} \end{pmatrix}_{4 \times 4} = \frac{1}{N} \left\langle \mathrm{tr}\,_{b4} \frac{1}{\mathcal{W} \otimes \mathbf{1} - \mathcal{H}} \right\rangle$
  where $\mathcal{W} = \mathrm{diag}(w, w, \bar{w}, \bar{w})$
- Similarly $\Sigma(\mathcal{W}) = \mathcal{R}(\mathcal{G}(\mathcal{W}))$ where $\mathcal{G}(\mathcal{W}) = (\mathcal{W} - \Sigma(\mathcal{W}))^{-1}$ are all 4 by 4 matrices

Elements of the construction (II)

- Flow of indices yields $\Sigma = \begin{pmatrix} 0 & \Sigma_{AA} & \Sigma_{A\bar{A}} & 0 \\ \Sigma_{BB} & 0 & 0 & \Sigma_{B\bar{B}} \\ \Sigma_{\bar{B}B} & 0 & 0 & \Sigma_{\bar{B}\bar{B}} \\ 0 & \Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} & 0 \end{pmatrix}$
- Above eight elements can be grouped
$$\begin{pmatrix} \Sigma_{AA} & \Sigma_{A\bar{A}} \\ \Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{AA}(\mathcal{G}_B) & \mathcal{R}_{A\bar{A}}(\mathcal{G}_B) \\ \mathcal{R}_{\bar{A}A}(\mathcal{G}_B) & \mathcal{R}_{\bar{A}\bar{A}}(\mathcal{G}_B) \end{pmatrix} = \mathcal{R}_A(\mathcal{G}_B)$$
$$\begin{pmatrix} \Sigma_{BB} & \Sigma_{B\bar{B}} \\ \Sigma_{\bar{B}B} & \Sigma_{\bar{B}\bar{B}} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{BB}(\mathcal{G}_A) & \mathcal{R}_{B\bar{B}}(\mathcal{G}_A) \\ \mathcal{R}_{\bar{B}B}(\mathcal{G}_A) & \mathcal{R}_{\bar{B}\bar{B}}(\mathcal{G}_A) \end{pmatrix} = \mathcal{R}_B(\mathcal{G}_A)$$
- Matrices $\mathcal{G}_A$ and $\mathcal{G}_B$ are unknown (alike g_A, g_B)
$$\mathcal{G}_A = \begin{pmatrix} \mathcal{G}_{12} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{2}2} & \mathcal{G}_{\bar{2}\bar{1}} \end{pmatrix} \mathcal{G}_B = \begin{pmatrix} \mathcal{G}_{21} & \mathcal{G}_{2\bar{2}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{2}} \end{pmatrix}$$

Elements of the construction (III)

- Note that $G_{AB}(z,\bar{z}) = \mathcal{G}_{MM}(\mathcal{Z}) = \frac{\mathcal{G}_{11}(\mathcal{W})}{w}$, where $M = AB$. It is possible iff
$$\Sigma_M \equiv \begin{pmatrix} \Sigma_{MM} & \Sigma_{M\bar{M}} \\ \Sigma_{\bar{M}M} & \Sigma_{\bar{M}\bar{M}} \end{pmatrix} =$$
$$\begin{pmatrix} \Sigma_{AA} & \sqrt{\frac{w}{\bar{w}}}\Sigma_{A\bar{A}} \\ \sqrt{\frac{\bar{w}}{w}}\Sigma_{\bar{A}A} & \Sigma_{\bar{A}\bar{A}} \end{pmatrix} \cdot \begin{pmatrix} \Sigma_{BB} & \sqrt{\frac{\bar{w}}{w}}\Sigma_{B\bar{B}} \\ \sqrt{\frac{w}{\bar{w}}}\Sigma_{\bar{B}B} & \Sigma_{\bar{B}\bar{B}} \end{pmatrix} \equiv \Sigma_A^L \Sigma_B^R$$
- Recalling that $\Sigma_A = \mathcal{R}_A(\mathcal{G}_B)$ and $\Sigma_A = \mathcal{R}_A(\mathcal{G}_B)$ we have $\mathcal{R}_M(\mathcal{G}_M) = [\mathcal{R}_A(\mathcal{G}_B)]^L \cdot [\mathcal{R}_B(\mathcal{G}_A)]^R$.
- Tedious calculations allow to close the construction
$$\mathcal{G}_A = \left[\mathcal{G}_M \cdot [\mathcal{R}_A(\mathcal{G}_B)]^L\right]^L \qquad \mathcal{G}_B = \left[[\mathcal{R}_B(\mathcal{G}_A)]^R \cdot \mathcal{G}_M\right]^R$$

Elements of the construction (IV)

- Formally, one can define now *two* matrix-valued S-transforms via equations
- $\mathcal{S}^{(LEFT)}(\mathcal{Y}) = \frac{1}{\mathcal{R}^L\left(\left[\mathcal{S}^{(LEFT)}(\mathcal{Y})\mathcal{Y}\right]^R\right)}$
- $\mathcal{S}^{(RIGHT)}(\mathcal{Y}) = \frac{1}{\mathcal{R}^R\left(\left[\mathcal{Y}\mathcal{S}^{(RIGHT)}(\mathcal{Y})\right]^L\right)}$
- Multiplication law reads $\left[\mathcal{S}_M^{(LEFT)}\left([\mathcal{R}(\mathcal{G})\mathcal{G}]^L\right)\right]^R = \left[\mathcal{S}_M^{(RIGHT)}\left([\mathcal{G}\mathcal{R}(\mathcal{G})]^R\right)\right]^L = \mathcal{S}_B^{(RIGHT)}(\mathcal{G}\mathcal{R}_M(\mathcal{G})) \cdot \mathcal{S}_A^{(LEFT)}(\mathcal{R}_M(\mathcal{G})\mathcal{G})$

Algorithm at work

1. Write down known $\mathcal{R}_A(\mathcal{G}_B)$ and $\mathcal{R}_B(\mathcal{G}_A)$, where $\mathcal{G}_A = \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix}$ and $\mathcal{G}_B = \begin{pmatrix} a' & i\bar{b}' \\ ib' & \bar{a}' \end{pmatrix}$
2. Modify $\mathcal{R}_A \to \mathcal{R}_A^L$, $\mathcal{R}_B \to \mathcal{R}_B^R$
3. $\mathcal{R}_M = \mathcal{R}_A^L \mathcal{R}_B^R$
4. Write down consistency conditions $(\mathcal{Z} - \mathcal{R}_M)\mathcal{G}_A^R = [\mathcal{R}_A(\mathcal{G}_B)]^L$ and $\mathcal{G}_B^L(\mathcal{Z} - \mathcal{R}_M) = [\mathcal{R}_B(\mathcal{G}_A)]^R$
5. Solve (4) for a, b, a', b' and read out $\mathcal{G}_M$

Note that if A and B are identical free ensembles, pair (3) reduces to one equation, since $a = a'$, $b = b'$.

Example 1: GUE times GUE

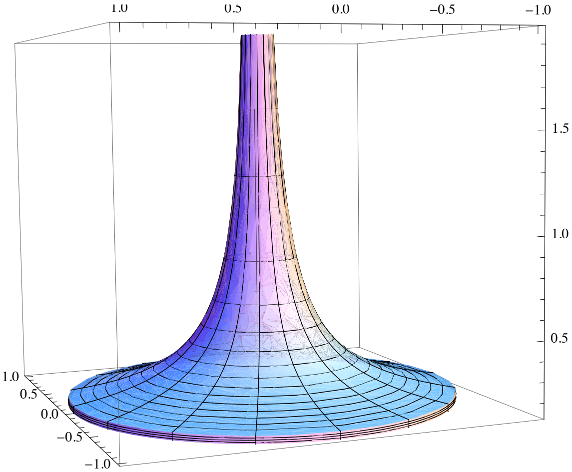
1. We ask for the spectrum of $H_A \cdot H_B$, where both ensembles are free (independent) GUE. Then for both ensembles $\mathcal{R}_A(\mathcal{G}) = \mathcal{R}_B(\mathcal{G}) = \mathcal{G}$, in analogy to $R(z) = z$.
2. $\mathcal{R}_A^L(\mathcal{G}) = \begin{pmatrix} a & i\bar{b}\sqrt{\frac{\bar{w}}{w}} \\ ib\sqrt{\frac{w}{\bar{w}}} & \bar{a} \end{pmatrix}$, $\mathcal{R}_B^R(\mathcal{G}) = \begin{pmatrix} a & i\bar{b}\sqrt{\frac{w}{\bar{w}}} \\ ib\sqrt{\frac{\bar{w}}{w}} & \bar{a} \end{pmatrix}$
3. $\mathcal{R}_M = \mathcal{R}_A^L \mathcal{R}_B^R$
4. Consistency conditions provide matrix equation for a, b.
5. Two solutions: $a = b = 0$ or $a = 0$, $|b|^2 = 1 - w\bar{w} = 1 - \sqrt{z\bar{z}}$
6. Holomorphic solution $G(z) = \frac{1}{z}$, nonholomorphic $G(z, \bar{z}) = \sqrt{\frac{\bar{z}}{z}}$, they match on boundary $|z| = 1$

Spectral density $\rho(z, \bar{z}) = \frac{1}{\pi}\partial_{\bar{z}} G(z, \bar{z}) = \frac{1}{2\pi}\frac{1}{\sqrt{x^2+y^2}}$ on unit disc.

So $W_A(0,1) \bigotimes W_B(0,1)$ is a "Halloween hat" law

[Girko,Vladimirowa;2009], [Burda,Janik,Wacław;2010]

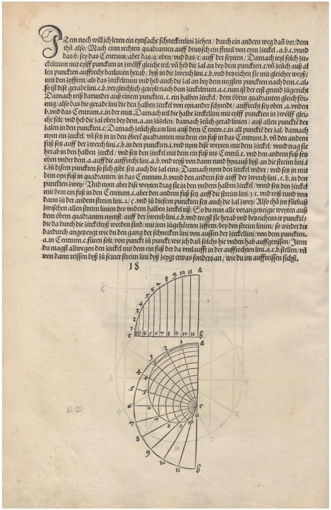
Example 1: Visualization



Note qualitative similarity to $N_a(0,1) \otimes N_b(0,1) = \frac{1}{\pi} K_0(x)$.

Example 2: Shifted Ginibre-Girko Ensemble times shifted Ginibre-Girko Ensemble

- We ask for the spectrum of $(1 + X_A)(1 + X_B)$, where both ensembles X_A, X_B are free (independent) GGE, i.e. spectrum of $1 + X$ is uniform unit disc centered at $x = 1, y = 0$.
- $\mathcal{R}_{1+X} \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix} = \begin{pmatrix} 1 & i\bar{b} \\ ib & 1 \end{pmatrix}$
- Algorithm yields the boundary and the spectral density
- Boundary belongs to the family of parametric curves appearing in non-hermitian diffusion, density involves the solution of biquadratic equation [Gudowska-Nowak,Janik,Jurkiewicz,MAN;2003] .
- Boundary reads $r = 1 + 2\cos\phi$, where $z = re^{i\phi}$.
- Curve known as Pascal limaçon (after Etienne Pascal (1588-1651), father of Blaise Pascal), but actually known already to Albrecht Dürer (Underweysung der Messung, 1525)



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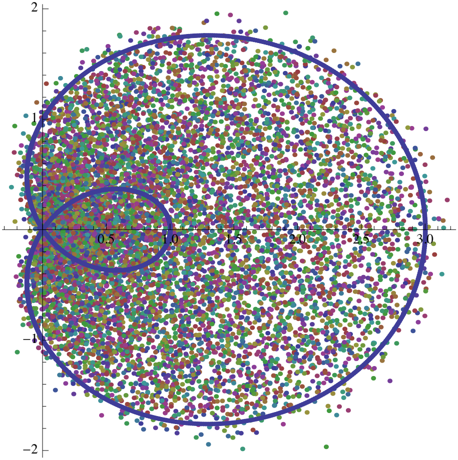
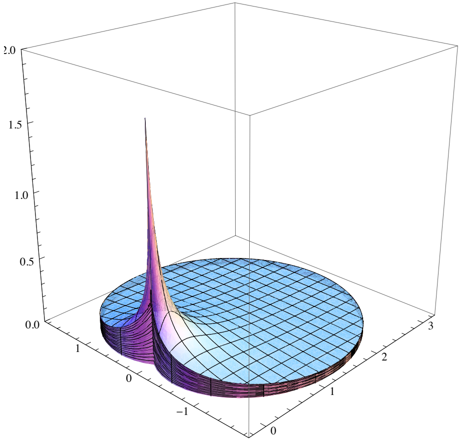
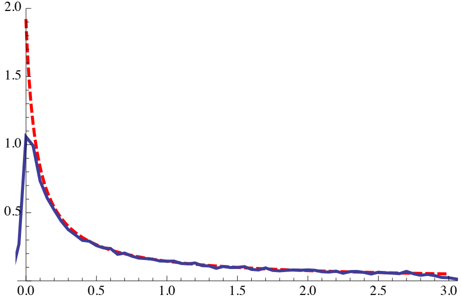
Limaçon: visualization five centuries later...



Limaçon in 3D



Limaçon: numerical crosscheck of spectral desnity at $y = 0$



Challenges

- Mathematical arguments why described construction is possible
- Geometric interpretation of the left and right rotation (connection to $SU(2)$), $X^L = UXU^\dagger$, $X^R = U^\dagger XU$, where $U = e^{i\frac{\sigma_3}{2}\frac{\phi}{2}}$.
- Less academic examples [Burda,Janik,MAN;2011b]

Summary

type of randomness	Addition law	Multiplication law
Random variables	$\surd$	$\surd$
Random matrices (1-d)	$\surd$	$\surd$
Random matrices (2-d)	$\surd$	$\surd$

"Quaternization" [Jarosz,Nowak;2004]

- For any hermitian H, knowing $G(z)$ and $R(z)$, we write down
  $\mathcal{G}(Q) = \frac{qG(q)-\bar{q}G\bar{q}}{q-\bar{q}}\mathbf{1}_2 - \frac{G(q)-G(\bar{q}}{q-\bar{q}}Q^\dagger$
  $\mathcal{R}(Q) = \frac{qR(q)-\bar{q}R\bar{q}}{q-\bar{q}}\mathbf{1}_2 - \frac{R(q)-R(\bar{q}}{q-\bar{q}}Q^\dagger$
  where $q, \bar{q}$ are eigenvalues of the quaternion Q. Note that $\mathcal{R}(\mathcal{G}(Q) + 1/Q) = Q$
- In analogy to $G_{tH}(z) = \frac{1}{t}G_H(\frac{z}{t})$ and $R_{tH} = tR_H(tz)$ for t real we have now for complex t
  $\mathcal{G}_{tX}(Q) = \mathcal{G}_X(\mathcal{T}^{-1}Q)\mathcal{T}^{-1}$ and $\mathcal{R}_{tX}(Q) = \mathcal{T}\mathcal{R}_X(Q\mathcal{T})$, where $\mathcal{T} = \mathrm{diag}(t, \bar{t})$
- Similar formulae for unitary ensembles [Jarosz, Goerlich;2005], in particular for CUE we get

$$\mathcal{R}_{CUE}\begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix} = \begin{pmatrix} 0 & i\bar{b}\frac{1-\sqrt{1-4|b|^2}}{2|b|^2} \\ ib\frac{1-\sqrt{1-4|b|^2}}{2|b|^2} & 0 \end{pmatrix}$$