Symmetries of Lévy Processes on compact quantum groups

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Bialgebras in Free Probability
ESI, Wien, April 2011

Outline

Compact Quantum Groups: definition

Definition (Woronowicz)

A **compact quantum group** is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : A \to A \otimes A$ is a unital, $*$-homomorphism which is coassociative, i.e.

$$(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$$

such that the quantum cancellation rules are satisfied

$$\overline{\mathrm{Lin}}((1 \otimes A)\Delta(A)) = \overline{\mathrm{Lin}}((A \otimes 1)\Delta(A)) = A \otimes A.$$

The Hopf $*$-algebra of "smooth" elements

A unital $*$-algebra $\mathcal{A}$ is called $*$-**bialgebra** if it is equipped with two $*$-homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (*coproduct*) and $\varepsilon : \mathcal{A} \to \mathbb{C}$ (*counit*) satisfying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \quad (\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}.$$

The Hopf $*$-algebra of "smooth" elements

A unital $*$-algebra $\mathcal{A}$ is called **$*$-bialgebra** if it is equipped with two $*$-homomorphisms $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (*coproduct*) and $\varepsilon : \mathcal{A} \to \mathbb{C}$ (*counit*) satisfying

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A $*$-bialgebra $\mathcal{A}$ is called a **Hopf $*$-algebra** if there exists a linear mapping $S : \mathcal{A} \to \mathcal{A}$ (*antipode*) such that $S \circ * \circ S \circ * = \mathrm{id}$ and

$$(S \otimes \mathrm{id}) \circ \Delta(a) = \varepsilon(a)\mathbf{1} = (\mathrm{id} \otimes S) \circ \Delta(a).$$

Unitary corepresentations

- n-**dimensional unitary corepresentation** of A :
  $U = (u_{jk})_{1 \le j,k \le n} \in M_n(\mathrm{A})$ a unitary such that for all $j, k = 1, \ldots, n$ we have
  $$\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}.$$
- Let $(U^{(s)})_{s \in \mathcal{I}}$ be a complete family of mutually inequivalent irreducible unitary correpresentations of A
- The algebra of the "smooth" elements of A is defined as
  $$\mathcal{A} = \mathrm{Lin}\{u_{jk}^{(s)}; s \in \mathcal{I}, 1 \le j,k \le n_s\},$$
  where n_s is the dimension of $u^{(s)}$.

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  where n_s is the dimension of $u^{(s)}$.

$\mathcal{A}$ is a dense $*$-subalgebra of A, which is a Hopf $*$-algebra with
$$\varepsilon(u_{jk}^{(s)}) = \delta_{jk} \quad \text{and} \quad S(u_{jk}^{(s)}) = (u_{kj}^{(s)})^*.$$

Example $SU_q(2)$

Let $q \in [-1,0) \cup (0,1]$. The quantum group $SU_q(2)$ is the C^*-algebra generated by the coefficients of the matrix

$$U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

with the relations on α and γ that ensure $UU^* = U^*U = 1$ and that the quantum determinant $D(U) = 1$ and with the comultiplication

$$\Delta(\alpha) = \alpha \otimes \alpha + \gamma \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Example $SU_q(2)$

The Hopf algebra $\mathcal{A}$ is given by

$$\mathcal{A} = *\text{-Alg}\{\alpha, \gamma\} = \text{Pol}(\alpha, \alpha^*, \gamma, \gamma^*),$$
$$\varepsilon(\alpha) = 1, \varepsilon(\gamma) = 0, \quad S(\alpha) = \alpha^*, S(\gamma) = -q\gamma.$$

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The Hopf algebra $\mathcal{A}$ is given by

$$\begin{aligned} &\mathcal{A} = *\text{-Alg}\{\alpha,\gamma\} = \text{Pol}(\alpha,\alpha^*,\gamma,\gamma^*), \\ &\varepsilon(\alpha)=1, \varepsilon(\gamma)=0, \quad S(\alpha)=\alpha^*, S(\gamma)=-q\gamma. \end{aligned}$$

On the other hand, $\mathcal{A} = \text{Lin}\{u^{(s)}_{jk}; s\in\mathcal{I}, 1\le j,k\le n_s\}$, where $\mathcal{I} = \frac{1}{2}\mathbb{N}$, $U^{(0)} = (\mathbf{1})$, $U^{(\frac{1}{2})} = U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$,

$$U^{(1)} = \begin{pmatrix} \alpha^2 & -q\sqrt{1+q^2}\gamma^*\alpha & q^2(\gamma^*)^2 \\ \sqrt{1+q^2}\gamma\alpha & 1-(1+q^2)\gamma^*\gamma & -q\sqrt{1+q^2}\alpha^*\gamma^* \\ \gamma^2 & \sqrt{1+q^2}\alpha^*\gamma & (\alpha^*)^2 \end{pmatrix},$$

etc.

Example $SU_q(N)$

The quantum group $SU_q(N)$ is generated by the matrix elements of $U = [u_{ij}]_{i,j=1,\ldots,N}$ satisfying the relations

$$
\begin{aligned}
u_{ij}u_{kj} &= q u_{kj}u_{ij} \quad \text{for } i<k, && (1)\\
u_{ij}u_{il} &= q u_{il}u_{ij} \quad \text{for } j<l, && (2)\\
u_{ij}u_{kl} &= u_{kl}u_{ij} \quad \text{for } i<k, j>l, && (3)\\
u_{ij}u_{kl} &= u_{kl}u_{ij} + q^{-1}(1-q^2)u_{il}u_{kj} \quad \text{for } i<k, j<l, && (4)
\end{aligned}
$$

with the additional requirement on the *quantum determinant*

$$D = D(U) := \sum_{\sigma\in S_n} (-q)^{i(\sigma)} u_{1,\sigma(1)}\cdots u_{n,\sigma(n)} = 1.$$

The involution is determined by the relation $UU^* = U^*U = \mathbf{1}$.

Example $SU_q(N)$

We have

$$\mathcal{A} = *\text{-Alg}\{u_{ij}; i,j = 1,\ldots,N\}$$

$$\Delta(u_{jk}) = \sum_{p=1}^{n} u_{jp} \otimes u_{pk}, \quad \varepsilon(u_{jk}) = \delta_{jk}, \quad S(u_{jk}) = u_{jk}^*.$$

The matrix U is a corepresenatation and the family of irreducible, inequivalent, unitary correpresetations is indexed by $(\frac{1}{2}\mathbb{N})^{N-1}$.

The Haar state

Notation: for $a \in \mathrm{A}$ and $\xi, \xi' \in \mathrm{A}'$

$$\begin{aligned} \xi \star \xi'(a) &= (\xi \otimes \xi')\Delta(a) \\ \xi \star a &= (\mathrm{id} \otimes \xi)\Delta(a) \\ a \star \xi &= (\xi \otimes \mathrm{id})\Delta(a) \end{aligned}$$

Theorem (Woronowicz)

Let (A, Δ) be a compact quantum group. There exists unique state (called the **Haar measure**) h on A such that

$$a \star h = h \star a = h(a)I, \quad a \in \mathrm{A}.$$

Note that, in general, h is not a trace!

Woronowicz characters

Theorem (Woronowicz)

Then there exists a unique family $(f_z)_{z\in\mathbb{C}}$ of linear multiplicative functionals on $\mathcal{A}$, called the **Woronowicz characters**, such that:

1. $f_z(\mathbf{1}) = 1$ for $z \in \mathbb{C}$ and $f_0 = \varepsilon$
2. $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is an entire holomorphic function.
3. $f_{z_1} \star f_{z_2} = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$.
4. $f_z(S(a)) = f_{-z}(a)$ and $f_{\bar{z}}(a^*) = \overline{f_{-z}(a)}$, for any $z \in \mathbb{C}$, $a \in \mathcal{A}$.
5. $S^2(a) = f_{-1} \star a \star f_1$ for $a \in \mathcal{A}$.
6. $h(ab) = h\left(b(f_1 \star a \star f_1)\right)$ for $a, b \in \mathcal{A}$.

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Example for $SU_q(2)$: $f_z(u_{jk}^{(s)}) = q^{z(j+k)}\delta_{jk}$

Decomposition of the antipode

Theorem (Woronowicz):

The closure $\overline{S}$ of the antipode S admits the "polar" decomposition:

$$\overline{S} = R \circ \tau_{\frac{i}{2}},$$

where

- $\tau_{\frac{i}{2}}$ is the analytic generator of a one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of the C^*-algebra A
- R is a linear antimultiplicative norm preserving involution, commuting with the adjoint, that commutes with the$(\tau_t)_{t\in\mathbb{R}}$, i.e. $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$.

From the proof:

$$\begin{aligned} \tau_t(a) &= f_{it} \star a \star f_{-it}, \\ R(a) &= S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}}) \end{aligned}$$

Noncommutative Lévy Processes

Let $\mathcal{A}$ be a $*$-bialgebra and let $(\mathcal{P}, \Phi)$ be a noncommutative probability space.

- a **random variable** on $\mathcal{A}$ over $(\mathcal{P}, \Phi)$ is a $*$-algebra homomorphism from $\mathcal{A}$ into the space $(\mathcal{P}, \Phi)$
- the **distribution** of the random variable $j : \mathcal{A} \to \mathcal{P}$ is the state $\varphi_j = \Phi \circ j$
- a **quantum stochastic process** on $\mathcal{A}$ is a family of random variables $(j_t)_{t \in J}$ on $\mathcal{A}$ indexed by a set J
- the **convolution product** of $j_1, j_2 : \mathcal{A} \to \mathcal{P}$ is the random variable $j_1 \star j_2 = m_{\mathcal{P}} \circ (j_1 \otimes j_2) \circ \Delta$, where $m_{\mathcal{P}}$ denotes the product in $\mathcal{P}$.

Noncommutative Lévy Processes

A quantum stochastic process $(j_{st})_{0\le s\le t\le T}$ $(T\in\mathbb{R}_+\cup\{\infty\})$ on a $*$-bialgebra $\mathcal{A}$ over $(\mathcal{P},\Phi)$ is called **Lévy process** if it satisfies:

- (**increment property**)

$$j_{rs}\star j_{st}=j_{rt}\quad\text{for all } 0\le r\le s\le t\le T$$

  and $j_{tt}=\varepsilon\mathbf{1}_{\mathcal{P}}$ for all $0\le t\le T$,
- the increments (j_{st}) are (tensor) **independent**, i.e. for disjoint intervals $(t_i,s_i]$

$$\Phi\big(j_{s_1t_1}(a_1)\ldots j_{s_nt_n}(a_n)\big)=\Phi\big(j_{s_1t_1}(a_1)\big)\ldots\Phi\big(j_{s_nt_n}(a_n)\big)$$

  and $[j_{s_i,t_i}(a_1),j_{s_j,t_j}(a_2)]=0$ for $i\neq j$,
- the increments (j_{st}) are **stationary**, i.e. $\varphi_{st}=\Phi\circ j_{st}$ depends only on $t-s$,
- (**weak continuity**) j_{st} converges to j_{ss} in distribution for $t\searrow s$.

The convolution semigroup and the generator of a NC Lévy process

The marginal distributions $\varphi_{s-t} := \varphi_{st} = \Phi \circ j_{st}$ of a Lévy process $(j_{st})_{0 \leq s \leq t}$ form a convolution semigroup of states, i.e.

- $\varphi_0 = \varepsilon$, $\varphi_s \star \varphi_t = \varphi_{s+t}$, $\lim_{t \to 0} \varphi_t(b) = \varepsilon(b)$ for all $b \in \mathcal{A}$,
- $\varphi_t(\mathbf{1}) = 1$, $\varphi_t(b^* b) \geq 0$ for all $b \in \mathcal{A}$ and $t \geq 0$.

There exists a unique functional $L : \mathcal{A} \to \mathbb{C}$, called the **generating functional**, such that

$$\varphi_t = \exp_\star tL \quad \text{and} \quad L = \frac{d}{dt}\varphi_t\Big|_{t=0}.$$

Lévy Processes and Markov semigroup

Given a convolution semigroup of states $(\varphi_t)_{t\geq 0}$, we can define a semigroup of operators

$$T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta, \quad t \geq 0.$$

Its **infinitesimal generator** $G : \mathcal{A} \to \mathcal{A}$ is the convolution operator associated to the generating functional L, i.e.

$$G(a) = (\mathrm{id} \otimes L) \circ \Delta(a) = L \star a.$$

Notation: $G = T_L$.

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Remark

$G : \mathcal{A} \to \mathcal{A}$ is a convolution operator if and only if
$\Delta \circ G = (\mathrm{id} \otimes G) \circ \Delta$.
In this case $L(a) = \varepsilon \circ G(a)$.

Characterisation of Generators of Convolution Semigroups

Theorem (Schoenberg correspondence):

The functional $L : \mathcal{A} \to \mathbb{C}$ is a generating functional of a convolution semigroup of states if and only if L is

- **hermitian**, i.e. $L(b^*) = \overline{L(b)}$,
- **conditionally positive**, i.e. $L(b^*b) \geq 0$ provided $\varepsilon(b) = 0$,
- and $L(\mathbf{1}) = 0$.

Lévy Processes and the Generators

noncommutative Lévy process $(j_{st})_{0\leq s\leq t}$		
$\updownarrow$		
convolution semigroup of states $(\varphi_t)_{t\geq 0}$	$\leftrightarrow$	semigroup of Markov operators $(T_t)_{t\geq 0}$
$\updownarrow$		$\updownarrow$
generating functional $L:\mathcal{A}\to\mathbb{C}$	$\leftrightarrow$	infinitesimal generator $T_L:\mathcal{A}\to\mathcal{A}$
$\updownarrow$		
hermitian, cond. positive $L:\mathcal{A}\to\mathbb{C}$, s.t. $L(\mathbf{1})=0$		

Lévy Processes and the Generators

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Aim of the project: study the noncommutative **geometry** of a quantum group via its Lévy processes

Ideas / Problems / Questions :

- Which processes (and their generators) give interesting information about the nc geometry?
- Are nc Brownian motions (i.e. gaussian generators) useful for that?
- What other conditions (symmetries) on the generators would be appropriate?

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- Extend the theory of Dirichlet forms associated to Markov semigroups and the construction of their derivations to the non-tracial case (cf. Cipriani& Sauvageot)

'Commutative' part of $SU_q(n)$

Let

$$\begin{aligned} K_1 &= \ker\varepsilon, \\ K_2 &= \mathrm{Lin}\,\{a_1a_2 : a_1, a_2 \in \ker\varepsilon\}, \\ K_n &= \mathrm{Lin}\,\{a_1a_2\ldots a_n : a_1, a_2, \ldots, a_n \in \ker\varepsilon\}, \\ K_\infty &= \bigcap_{n\geq 1} K_n. \end{aligned}$$

'Commutative' part of $SU_q(n)$

Description of K_∞ for $SU_q(n)$

- $u_{ij}, u_{ij}^* \in K_\infty$ for $i \neq j$
- $u_{ii}u_{jj} = u_{jj}u_{ii}$, $u_{ii}u_{jj}^* = u_{jj}^*u_{ii}$ (modulo K_∞, for $i \neq j$)
- $u_{jj}u_{jj}^* = u_{jj}^*u_{jj} = 1$ (modulo K_∞)
- $u_{11} \ldots u_{nn} = 1$ (modulo K_∞)

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- $u_{11} \ldots u_{nn} = 1$ (modulo K_∞)

Theorem

The ideal K_∞ is also a coideal in $\mathcal{A}$, $\mathcal{A}/K_\infty$ is a $*$-bialgebra and

$$\mathcal{A}/K_\infty \cong \mathbb{C}(\mathbb{T}^{n-1}).$$

All processes for which $L|_{K_\infty} = 0$ are isomorphic to processes on the $(n-1)$-dimensional torus.

'Commutative' part of $SU_q(n)$

Definition

A generator L is called a **Gaussian** generator if $L|_{K_3} = 0$.

Observation

The gaussian processes on $SU_q(n)$ encode the structure of $(n-1)$-dimensional torus, i.e. they give no information on the **noncommutative** geometry of $SU_q(n)$.

For $SU_q(2)$ this was shown by
M. Schürmann and M. Skeide'1998.

Symmetric generators

We shall consider the inner product induced by the Haar state h

$$\langle a, b \rangle := h(a^* b).$$

Recall: each generator L of a Lévy process induces the operator

$$T_L(a) = L \star a = (\mathrm{id} \otimes L) \circ \Delta(a), \quad a \in \mathcal{A}.$$

Proposition

Each operator $T_L : \mathcal{A} \to \mathcal{A}$ admits unique adjoint, i.e. there exists a unique linear map $T_L^\star : \mathcal{A} \to \mathcal{A}$ such that

$$h\big(a^* T_L(b)\big) = h\big(T_L^\star(a)^* b\big)$$

for all $a, b \in \mathcal{A}$.

Symmetric generators on quantum groups

We say that a generating functional L is symmetric if the operator T_L is self-adjoint, i.e. if

$$h\big(a^* T_L(b)\big) = h\big(T_L(a)^* b\big), \quad a, b \in \mathcal{A}.$$

($\rightarrow$ **GNS**-symmetry).

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Question:

For which generators L of Lévy processes the operator T_L is self-adjoint?

Symmetric generators

One can show that

$$T_L^\star = T_{L^\# \circ S}, \quad \text{where } L^\#(a) = \overline{L(a^*)},$$

therefore, if L is hermitian, then $L^\# = L$.

Proposition: $T_L = T_L^\star \quad \text{iff} \quad L = L \circ S$

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Proposition: $T_L = T_L^\star$ iff $L = L \circ S$

Remark

A hermitian functional L is symmetric iff the matrices $L(U^{(s)})$ are hermitian:

$$S(u_{jk}^{(s)}) = (u_{kj}^{(s)})^* \Rightarrow L(u_{jk}^{(s)}) = (L \circ S)(u_{jk}^{(s)}) = L((u_{kj}^{(s)})^*) = \overline{L(u_{kj}^{(s)})}.$$

Symmetric generators

Examples

Let

$$L(u_{jk}^{(s)}) = c_s \delta_{jk}.$$

Then L is symmetric for any family $(c_s)_{s \in \mathcal{I}}$ of real numbers.

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Example in case of $SU_q(2)$

If L is symmetric, then $L(u_{jk}^{(s)}) = c_{s,j}\delta_{jk}$ with $c_{j,s} \in \mathbb{R}$.
If, moreover, L is hermitian, then $c_{j,s} = c_{-j,s}$.

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(follows from $S^2(u_{jk}^{(s)}) = q^{2(j-k)} u_{jk}^{(s)}$ and $(u_{jj}^{(s)})^* = u_{-j,-j}^{(s)}$)

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Problem

For which c_s, L is conditionally positive?

Symmetric generators: positivity

Problem: for which c_s is $L\mathcal{A} \to \mathbb{C}$ with $L(u_{jk}^{(s)}) = c_s \delta_{jk}$ is cond. positive?

Symmetric generators: positivity

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Idea: use the subalgebra spanned by the characters (traces) of irreducible unitary corepresentations.

Symmetric generators: positivity

for $SU_q(2)$ (works more generally for q-deformations $\mathbb{G}_g$ of compact simple Lie groups)

- $a \in \mathcal{A}$ positive $\Rightarrow \varphi_a(b) := h\left((f_{-\frac{1}{2}} \star a \star f_{-\frac{1}{2}})b\right)$ positive
- The subalgebra $\mathcal{A}_0$ spanned by traces of the coreps does not depend on q

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- The subalgebra $\mathcal{A}_0$ spanned by traces of the coreps does not depend on q
- If $a = \sum_s n_s^2 a_s \chi_s \in \mathcal{A}_0$ (with $\chi_s = \frac{1}{n_s}\sum_{k=-s}^{s} u_{kk}^{(s)}$), then

$$\varphi_a(u_{jk}^{(s)}) = \frac{a_s n_s}{D_s}\delta_{jk},$$

where D_s is the quantum dimension of $U^{(s)}$, $D_s = q^{-2s}\frac{1-q^{2s}}{1-q^2}$.

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  where D_s is the quantum dimension of $U^{(s)}$, $D_s = q^{-2s}\frac{1-q^{2s}}{1-q^2}$.
- If (a_s) are s.t. a is positive for $SU(2)$ (classical group), then φ_a is positive and $L := \varphi_a - \varphi_a(\mathbf{1})\varepsilon$ is conditionally positive.

KMS-symmetry

Definition

Let $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous group of automorphisms of a C^*-algebra A and $\beta \in \mathbb{R}$. A state ω is said to be a (α, β)-**KMS state** if it is α-invariant and if the following KMS-condition holds true:

$$\omega(a\alpha_{i\beta}(b)) = \omega(ba)$$

for all a, b in a norm dense, α-invariant $*$-subalgebra of A.

- KMS states corresponding to $\beta = 0$ are just the traces over A.
- KMS states $\sim$ equilibrium of quantum statistical systems

KMS-symmetry

Let $\alpha := \{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous group of automorphisms of a C^*-algebra A and ω be a fixed (α, β)-KMS state, for some $\beta \in \mathbb{R}$.

Definition of KMS-symmetry

An operator $\Phi : A \to A$ is said to be (α, β)-**KMS symmetric** with respect to ω if

$$\omega\big(b\,\Phi(a)\big) = \omega\big(\alpha_{-\frac{i\beta}{2}}(a)\,\Phi(\alpha_{+\frac{i\beta}{2}}(b)\big)$$

for all a, b in a norm dense, α-invariant $*$-subalgebra B of A.

KMS-state on quantum groups

Theorem (Woronowicz):

The formula

$$\sigma_t(a) = f_{it} \star a \star f_{it}$$

defines a one parameter group of modular automorphisms of $\mathcal{A}$ and the Haar measure h is a $(\sigma, -1)$-KMS state:

$$h(ab) = h(b(f_1 \star a \star f_1)) = h(b\sigma_i(a)), \quad a, b \in \mathcal{A}.$$

Question:

For which generators L of Lévy processes the operator T_L is KMS-symmetric?

KMS-symmetry

The condition of the KMS-symmetry of an operator T_L can be expressed as a relation between T_L and $T_L^\star$:

$$h\big(T_L^\star(a)^* b\big) = h\left(a^* T_L(b)\right) = h\big((\sigma_{-\frac{i}{2}} \circ T_L \circ \sigma_{\frac{i}{2}})(a)^* b\big).$$

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Using $T_L = L \star a$, $T_L^\star = \left(L^\# \circ S\right) \star a$ and $\sigma_t(a) = f_{it} \star a \star f_{it}$, we have

$$L \star a = f_{-\frac{1}{2}} \star \left(L^\# \circ S\right) \star f_{\frac{1}{2}} \star a.$$

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If L is hermitian, this reduces to

$$L(a) = \varepsilon \circ (L \star a) = (L \circ S)(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}})$$

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Recall: $\overline{S} = R \circ \tau_{\frac{i}{2}}$, $R(a) = S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}})$

KMS-symmetry

Proposition

Let $L \in \mathcal{A}'$ be hermitian. Then

$$T_L \text{ is self-adjoint} \quad \text{iff} \quad L \circ S = L.$$

KMS-symmetry

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$$\begin{aligned} T_L \text{ is self-adjoint} \quad &\text{iff} \quad L \circ S = L. \\ T_L \text{ is KMS-symmetric} \quad &\text{iff} \quad L \circ R = L. \end{aligned}$$

KMS-symmetry

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Let $L \in \mathcal{A}'$ be hermitian. Then

$$T_L \text{ is self-adjoint} \quad \text{iff} \quad L \circ S = L.$$
$$T_L \text{ is KMS-symmetric} \quad \text{iff} \quad L \circ R = L.$$

Remark

If L is a generating functional, then

- $L + L \circ R$ is a generating functional (it is conditionally positive!)
- $T_{L+L\circ R}$ is KMS-symmetric.

Relations between symmetry and KMS-symmetry

Relations between symmetry and KMS-symmetry

Theorem

For $L \in \mathcal{A}'$ the following are equivalent:

1. T_L commutes with the modular group σ: $T_L \circ \sigma_t = \sigma_t \circ T_L$,
2. L commutes with the Woronowicz characters: $L \star f_z = f_z \star L$ for $z \in \mathbb{C}$,
3. L is invariant under $\tau_{\frac{i}{2}}$: i.e. $L \circ \tau_{\frac{i}{2}} = L$.

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Remark

- If L is symmetric, then L commutes with the Woronowicz characters and is KMS-symmetric.
- If the algebra is of Kac type ($S^2 = \mathrm{id}$), then $R = S$ and the symmetric and KMS-symmetric generators coincide.

One more idea: ad-Invariance

Definition

The *adjoint action* of a Hopf algebra is defined by $\mathrm{ad}:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$,

$$\mathrm{ad}(a)=a_{(1)}S(a_{(3)})\otimes a_{(2)},\qquad a\in\mathcal{A}.$$

Remarks

- The adjoint action is a left corepresentation, i.e. we have

$$\begin{aligned}(\mathrm{id}\otimes\mathrm{ad})\circ\mathrm{ad} &= (\Delta\otimes\mathrm{id})\circ\mathrm{ad},\\ (\varepsilon\otimes\mathrm{id})\circ\mathrm{ad} &= \mathrm{id}.\end{aligned}$$

- ad is not an algebra homomorphism.

ad-Invariance

Definition

A linear map $T \in \mathrm{Lin}(\mathcal{A})$ is called ad-*invariant*, if

$$(\mathrm{id} \otimes T) \circ \mathrm{ad} = \mathrm{ad} \circ T.$$

A linear functional $L \in \mathcal{A}'$ is called ad-*invariant*, if

$$(\mathrm{id} \otimes L) \circ \mathrm{ad} = L\mathbf{1}_{\mathcal{A}}.$$

Remarks

- The counit ε and the Haar state h are ad-invariant.
- For $L \in \mathcal{A}'$, T_L is ad-invariant if and only if L is ad-invariant.
- If $L, L' \in \mathcal{A}'$ are ad-invariant then $L \star L'$ is ad-invariant.

Question.

ad-Invariance

Denote by $\mathrm{ad}_h \in \mathrm{Lin}(\mathcal{A})$ the linear map given by

$$\mathrm{ad}_h = (h \otimes \mathrm{id}) \circ \mathrm{ad}.$$

Properties

- $L \circ \mathrm{ad}_h$ is ad-invariant for all $L \in \mathcal{A}'$.
- $L \in \mathcal{A}'$ is ad-invariant if and only if $L = L \circ \mathrm{ad}_h$.
- A functional L is ad-invariant iff it is of the form $L(u_{jk}^{(s)}) = c_s \delta_{jk}$ for some $c_s \in \mathbb{C}$.

ad-Invariance

Remarks

- If L is ad-invariant and hermitian, then L is symmetric if and only if $c_s \in \mathbb{R}$ for all $s \in \mathcal{I}$.
- $L \to L \circ \mathrm{ad}_h$ does not preserve the hermitianity, neither positivity!

Example

Indeed, let $L : SU_q(2) \to \mathbb{C}$

$$L(\alpha) := e^{it}, \quad L(\alpha^*) := e^{-it}$$

and zero otherwise.
Then $L_{\mathrm{ad}} = L \circ \mathrm{ad}_h$ is ad-invariant but not hermitian.

From Lévy Processes to Dirichlet Forms...

What next?

$$\text{Lévy process} \quad \longrightarrow \quad \text{Markov semigroup}$$

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\end{array}
$$

From Lévy Processes to Dirichlet Forms...

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$$\begin{array}{ccc} \text{Lévy process} & \longrightarrow & \text{Markov semigroup} \\ & \longrightarrow & \text{Dirichlet form } \mathcal{E} \\ & \longrightarrow & \text{derivation } \partial \\ & \longrightarrow & \text{Dirac operator } D \end{array}$$

Remark

The study of Dirichlet forms in the noncommutative setting of a C^* or von Neumann algebra (tracial case): S. Albeverio, R. Hoegh-Krohn, J.-L. Sauvageot, E.B. Davies, J.M. Lindsay, S. Goldstein, F. Cipriani, etc.

From Lévy Processs to Dirichlet Forms...

Theorem

Let $(T_t)_{t\geq 0}$ be the Markov semigroup of a Lévy process on $\mathcal{A}$ with generating functional L.

(a) $(T_t)_{t\geq 0}$ satisfies the *quantum detailed balance* condition, i.e. we have

$$h\big(aT_t(b)\big) = h\big(T_t(a)b\big)$$

for all $t \geq 0$ and all $a, b \in \mathcal{A}$, if and only if L is symmetric.

(b) $(T_t)_{t\geq 0}$ is KMS-symmetric if and only if L is KMS-symmetric.

From Lévy Processes to Dirichlet Forms...

Let A be a coamenable compact quantum group and let L be a generator of a Lévy process with the semigroup of states $(\varphi_t)_{t\geq 0}$ on $\mathcal{A}$. Then:

- There exists a one-to-one correspondence between weakly continuous semigroup of states on $\mathcal{A}$ and weakly continuous semigroup of states on A.
- The operator semigroup $T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta$ extends to the strongly continuous operator semigroup on A.
- There exists a closed, densely defined operator in A, which is the closure of T_L.

From Lévy Process to Dirichlet Form...

If L is a generating functional, then we can define

$$\mathcal{E}[a] = h\big(a^* \sigma_{-\frac{i}{2}}(T_L(a))\big), \qquad a \in \mathcal{A}.$$

From Lévy Process to Dirichlet Form...

If L is a generating functional, then we can define

$$\mathcal{E}[a] = h\big(a^*\sigma_{-\frac{i}{2}}(T_L(a))\big), \qquad a \in \mathcal{A}.$$

Example: $L(u_{jk}^{(s)}) = c_s\delta_{jk}$ on $SU_q(2)$

Then

$$\mathcal{E}[u_{jk}^{(s)}] = h\big((u_{jk}^{(s)})^*\sigma_{-\frac{i}{2}}(T_L(u_{jk}^{(s)}))\big) = \frac{c_s}{D_s}q^{k-j}.$$

Further directions...

Problems

- Find interesting explicit examples of symmetric or KMS-symmetric generators on $SU_q(n)$.
- Construct the related derivations and Dirac operators (need to extend Cipriani & Sauvaveot's construction to the non-tracial case).

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