Simplified quantum $E(2)$ group.

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$\mathbb{E}(2)$ versus simplified $E(2)$

Double covering:

$$\mathbb{E}(2) \ni \begin{pmatrix} v\,, & n \\ 0\,, & v^{-1} \end{pmatrix} \longmapsto \begin{pmatrix} u\,, & m \\ 0\,, & 1 \end{pmatrix} \in E(2),$$

where $u = v^2$ and $m = vn$.

COMMUTATION RELATIONS

v is unitary	u is unitary
n is normal	$mm^* = q^2 m^* m$
$\mathrm{Sp}\,\lvert n\rvert \subset q^{\mathbb{Z}} \cup \{0\}$	$\mathrm{Sp}\,\lvert m\rvert \subset q^{\mathbb{Z}} \cup \{0\}$
$vnv^* = qn$	$umu^* = q^2 m$

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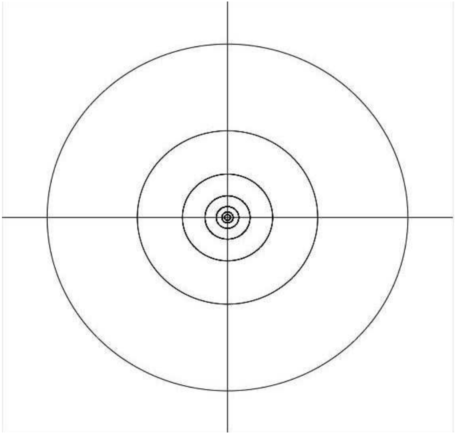
Quantised complex plane

Let q be the deformation parameter: $0 < q < 1$. We set

$$\overline{\mathbb{C}}^q = \{\gamma : |\gamma| \in q^{\mathbb{Z}} \cup \{0\}\}$$



Special function

We shall use the following function:

$$\overline{\mathbb{C}}^q \ni \gamma \longmapsto F_q(\gamma) \in S^1$$

$$F_q(\gamma) = \begin{cases} \displaystyle\prod_{k=0}^{\infty} \frac{1+q^{2k}\overline{\gamma}}{1+q^{2k}\gamma} & \text{for } \gamma \neq -q^{-2k}, \\ -1 & \text{otherwise.} \end{cases}$$

Then F_q is a continuous function: $F_q \in \mathcal{C}(\overline{\mathbb{C}}^q)$.

Commutation relations versus Operator domains

Let $CR(r, s, \dots, w)$ be commutation relations imposed on N operators (symbols) $r, s, \dots, w$. Then for any Hilbert space $\mathcal{H}$ one may consider the set of all N-tuples of closed operators satisfying the relations:

$$\mathcal{D}_{\mathcal{H}}^{CR} = \left\{(r, s \dots, w) \in \mathcal{C}_{\mathcal{H}}^{N} : CR(r, s, \dots, w)\right\}$$

We say that $\mathcal{D}^{CR}$ is an operator domain corresponding to the commutation relations CR. Mathematically operator domains are concrete topological W^*-categories.

Concrete topological W^*-categories

- Objects are N-tuples of operators. Each object r is anchored to a Hilbert space where the operators act. For each Hilbert space $\mathcal{H}$ the objects anchored to $\mathcal{H}$ form a set $\mathcal{D}_{\mathcal{H}}$.
- Morphisms are intertwining operators. For any $r \in \mathcal{D}_{\mathcal{H}_1}$ and $s \in \mathcal{D}_{\mathcal{H}_2}$, $\mathrm{Mor}(r, s)$ is a weakly closed linear subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$. Composition of morphisms is the composition of intertwinners. Moreover

$$(m \in \mathrm{Mor}(r, s)) \Rightarrow (m^* \in \mathrm{Mor}(s, r))$$

- $\mathcal{D}$ is complete and closed with respect to the natural topology.

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- $\mathcal{D}$ is complete and closed with respect to the natural topology.

Completeness

In brief, the concrete W^*-category $\mathcal{D}$ is complete if it is closed under unitary equivalence, direct sums and passing to a subobject. More precisely the following statement holds:

Consider any family of objects $r_\lambda \in \mathcal{D}_{\mathcal{H}_\lambda}$ and any family of operators $m_\lambda \in B(\mathcal{H}, \mathcal{H}_\lambda)$ (where $\mathcal{H}$ is a Hilbert space and λ runs over an index set Λ) such that

$$\bigcap_{\lambda \in \Lambda} \ker m_\lambda = \{0\} \quad \text{and} \quad m_{\lambda'} m_\lambda^* \in \mathrm{Mor}(r_\lambda, r_{\lambda'})$$

for all $\lambda, \lambda' \in \Lambda$. Then there exists unique $r \in \mathcal{D}_\mathcal{H}$ such that $m_\lambda \in \mathrm{Mor}(r, r_\lambda)$ for all $\lambda \in \Lambda$.

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Natural topology

Definition (*z*-transform)

For any closed operator r acting on a Hilbert space $\mathcal{H}$ we set

$$z_r = r\left(I + r^* r\right)^{-1/2}.$$

We say that a sequence $r_n \in \mathcal{C}_{\mathcal{H}}$ of closed operators is converging to $r \in \mathcal{C}_{\mathcal{H}}$ if

$$\lim_{n\to\infty} \left\| z_{r_n} x - z_r x \right\| = 0$$
$$\lim_{n\to\infty} \left\| z_{r_n}^* x - z_r^* x \right\| = 0$$

for any $x \in \mathcal{H}$

With this topology, $\mathcal{D}_{\mathcal{H}}^{CR}$ is closed in $\mathcal{C}_{\mathcal{H}}^{N}$ for each Hilbert space $\mathcal{H}$.

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With this topology, $\mathcal{D}^{CR}_\mathcal{H}$ is closed in $\mathcal{C}^N_\mathcal{H}$ for each Hilbert space $\mathcal{H}$.

Operator functions

In particular one may consider the empty set of relations imposed on one symbol. The operator domain corresponding to such relations coincides with the category $\mathcal{C}$ of closed operators.

Definition

We say that F is an operator function defined on an operator domain $\mathcal{D}$ if for any Hilbert space $\mathcal{H}$,

$$\mathcal{D}_{\mathcal{H}} \ni r \longrightarrow F(r) \in \mathcal{C}_{\mathcal{H}}$$

is a continuous mapping such that

$$\mathrm{Mor}(r, s) \subset \mathrm{Mor}(F(r), F(s))$$

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Operator functions

In brief

F is an operator function defined on $\mathcal{D}$ if F is a continuous functor from $\mathcal{D}$ into $\mathcal{C}$ that do not change the anchor Hilbert space and act trivially on morphisms.

We have:

$$F(uru^*) = uF(r)u^*$$
$$F(r \oplus s) = F(r) \oplus F(s)$$

In the first formula $r \in \mathcal{D}_{\mathcal{H}}$, u is an unitary operator acting form $\mathcal{H}$ onto $\mathcal{H}'$ and $uru^* \in \mathcal{D}_{\mathcal{H}'}$ is the unique object such that $u \in \mathrm{Mor}(r, uru^*)$.

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Important concrete topological W^*-categories

- Category $\mathrm{Rep}\,(A, -)$ of representations of a C^*-algebra A

Objects anchored to $\mathcal{H}$ are representations of the C^*-algebra A acting on $\mathcal{H}$. A sequence $\pi_n \in \mathrm{Rep}\,(A, \mathcal{H})$ is converging to $\pi \in \mathrm{Rep}\,(A, \mathcal{H})$ if for each $a \in A$, the sequence $\pi_n(a) \to \pi(a)$ in strong operator topology.

- Category $\mathcal{U}ni(\mathcal{K} \otimes -)$, where $\mathcal{K}$ is a fixed Hilbert space.

Objects anchored to $\mathcal{H}$ are unitary operators acting on $\mathcal{K} \otimes \mathcal{H}$. The topology on $\mathcal{U}ni(\mathcal{K} \otimes \mathcal{H})$ is the topology of *-strong convergence.

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Quantum spaces and continuous maps

Quantum spaces	Topological W^*-categories
Operator mappings	Functors that preserve the anchor Hilbert space and act trivially on morphisms

Operators domains relevant to old quantum $E(2)$.

$$\mathcal{E}_{\mathcal{H}} = \left\{ R \in \mathcal{C}_H : \begin{array}{l} R-\text{normal} \\ \operatorname{Sp} R \subset \overline{\mathbb{C}}^q \end{array} \right\}.$$

$$\mathcal{E}^2_{\mathcal{H}} = \left\{ (R,S) \in \mathcal{E}_{\mathcal{H}} \times \mathcal{E}_{\mathcal{H}} : \begin{array}{l} SR = q^2 RS \\ SR^* = R^* S \end{array} \right\}$$

Theorem

The closure of the sum is an operator mapping from $\mathcal{E}^2$ into $\mathcal{E}$:

$$\mathcal{E}^2_{\mathcal{H}} \ni (R,S) \longmapsto R \dot{+} S \in \mathcal{E}_{\mathcal{H}}.$$

$$\begin{aligned} R \dot{+} S &= F_q(R^{-1}S)\, R\, F_q(R^{-1}S)^*. \\ F_q(R \dot{+} S) &= F_q(R) F_q(S). \end{aligned} \tag{1}$$

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Exponential function equation

Theorem (Old EE Thm)

Let $\mathcal{K}$ be a Hilbert space and

$$\mathcal{E}_{\mathcal{H}} \ni R \longmapsto F(R) \in \mathcal{U}ni(\mathcal{K} \otimes \mathcal{H})$$

be an operator mapping. Then the following conditions are equivalent:

1. $$F(R \dot{+} S) = F(R)F(S) \tag{2}$$

   for any $(R, S) \in \mathcal{E}_{\mathcal{H}}^2$.

2. *There exists $T \in \mathcal{E}_{\mathcal{K}}$ such that*

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Then for any Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ we have:

$$\mathcal{F}_{\mathcal{K}}{}^* \otimes \mathcal{F}_{\mathcal{H}} \subset \mathcal{E}_{\mathcal{K}\otimes\mathcal{H}}, \qquad \mathcal{F}^*_{\mathcal{K}} \otimes \mathcal{F}^2_{\mathcal{H}} \subset \mathcal{E}^2_{\mathcal{K}\otimes\mathcal{H}},$$

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Operator mapping $\dot{+} : \mathcal{F}^2 \longrightarrow \mathcal{F}$.

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The closure of the sum is an operator mapping from $\mathcal{F}^2$ into $\mathcal{F}$:

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$$r \dot{+} s = F_q(r^{-1}s)\, r\, F_q(r^{-1}s)^*.$$

Proof: If $r = 0$ then the statement is obvious. Therefore we may assume that $\ker r = \{0\}$. Let $R = m \otimes r$ and $S = m \otimes s$, where $m \in {\mathcal{F}_K}^*$ with $\ker m = \{0\}$. Then $(R, S) \in \mathcal{E}_{K \otimes H}$ and we obtain:

$$m \otimes r \dot{+} m \otimes s = F_q(I \otimes r^{-1}s)(m \otimes r)F_q(I \otimes r^{-1}s)^*,$$

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Exponential function equation

Theorem (New EF Thm)

Let $\mathcal{K}$ be a Hilbert space and

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1. $$F(r \dot{+} s) = F(r)F(s)$$

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Proof

Formula (1) shows immediately that condition 2 implies condition 1. We shall prove the converse. Let $\mathcal{K}'$ be a Hilbert space and $m \in \mathcal{F}_{\mathcal{K}'}$ with $\ker m = \{0\}$. Then $m \otimes R \in \mathcal{F}_{\mathcal{K}' \otimes \mathcal{H}}$ for any $R \in \mathcal{E}_{\mathcal{H}}$ and we may consider the operator mapping

$$\mathcal{E}_{\mathcal{H}} \ni R \longmapsto F(m \otimes R) \in \mathcal{U}ni(\mathcal{K} \otimes \mathcal{K}' \otimes \mathcal{H})$$

By condition 1, this operator mapping fulfils the exponential equation (2). Therefore there exists $T \in \mathcal{E}_{\mathcal{K} \otimes \mathcal{K}'}$ such that

$$F(m \otimes R) = F_q(T \otimes R)$$

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$$\mathcal{E}_{\mathcal{H}} \ni R \longmapsto F(m \otimes R) \in \mathcal{U}ni(\mathcal{K} \otimes \mathcal{K}' \otimes \mathcal{H})$$

By condition 1, this operator mapping fulfils the exponential equation (2). Therefore there exists $T \in \mathcal{E}_{\mathcal{K} \otimes \mathcal{K}'}$ such that

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Operator $m \otimes m$ commutes with the flip $\Sigma_{\mathcal{K}'}$. Therefore $T \otimes m$ commutes with $I_{\mathcal{K}} \otimes \Sigma_{\mathcal{K}'}$. It implies that $T = t \otimes m$, where t is an operator acting on $\mathcal{K}$. One can easily verify that $t \in \mathcal{F}_{\mathcal{K}}{}^*$. Now our formula takes the form

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The domain $\mathcal{G}$ introduced below plays the role of quantum space $E(2)$. For any Hilbert space $\mathcal{H}$ we set

$$\mathcal{G}_{\mathcal{H}} = \left\{(u, m) \in \mathcal{U}ni(\mathcal{H}) \times \mathcal{F}_{\mathcal{H}} : umu^* = q^2 m\right\}$$

Then $\mathcal{G}$ is a closed complete operator domain.

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Unitary representations

For abelian locally compact groups we are interested in the space of characters. For non-abelian groups we look for strongly continuous unitary representations. For quantum groups the two notions unify.

Definition

Let $\mathcal{K}$ be a Hilbert space and

$$\mathcal{G}_{\mathcal{H}} \ni g \longmapsto U(g) \in \mathcal{U}ni(\mathcal{K} \otimes \mathcal{H})$$

be an operator mapping. We say that U is a unitary representation of quantum $E(2)$ acting on $\mathcal{K}$ if

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Description of unitary representations

Our aim is to find all unitary representations of quantum $E(2)$. The are labelled by the operator domain $\hat{\mathcal{G}}$. For any Hilbert space $\mathcal{K}$ we set

$$\hat{\mathcal{G}}_{\mathcal{K}} = \left\{ (\hat{N}, \hat{m}) : \begin{array}{c} \hat{N} \text{ is a selfadjoint operator} \\ \text{acting on } \mathcal{K}, \mathrm{Sp}\, N \subset \mathbb{Z}, \\ \hat{m} \in {\mathcal{F}_{\mathcal{K}}}^*, \ \hat{N}\hat{m} = \hat{m}(\hat{N} + I) \end{array} \right\}$$

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Proof (cont.)

Take $\mathcal{H} = \mathcal{H}' = \mathbb{C}$, $g = (\alpha, 0)$ *and* $g' = (\beta, 0)$, *where* $\alpha, \beta \in S^1$. *Then* $g \bigoplus g' = (\alpha\beta, 0)$ *and the character equation shows that*

$$U(\alpha\beta, 0) = U(\alpha, 0)U(\beta, 0)$$

Therefore $U(\alpha, 0) = \alpha^{\hat{N}}$, *where* $\hat{N}$ *is a selfadjoint operator acting on* $\mathcal{K}$ *with integer spectrum.*

Take $\mathcal{H}' = \mathbb{C}$, $g = (u, m)$ *and* $g' = (\alpha, 0)$, *where* $\alpha \in S^1$. *Then* $g \bigoplus g' = (\alpha u, m)$ *and the character equation shows that*

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Let $\Phi(u, m) = U(u, m)(I_{\mathcal{K}} \otimes u)^{-\hat{N} \otimes I_{\mathcal{H}}}$. *Then*

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$$\Phi(u, m) = \Phi((\mathrm{Phase}\, m)^2, m) = F(m)$$

$\Phi(u, m)$ *does not depend on* u. *Notice that* $F(0) = I_{\mathcal{K}}$

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Using the character equation in full generality we have:

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Proof (cont.)

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To end the proof we use the established earlier formula:

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$$\alpha^{\hat{N}} \hat{m} \alpha^{-\hat{N}} = \alpha \hat{m}$$

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Group structure on $\hat{\mathcal{G}}$.

Let $\hat{g} \in \hat{\mathcal{G}}_{\mathcal{K}}$ and $\hat{g}' \in \hat{\mathcal{G}}_{\mathcal{K}'}$. Then

$$\mathcal{G}_{\mathcal{H}} \ni g \longmapsto U(\hat{g}', g)_{23} U(\hat{g}, g)_{13} \in \mathcal{U}ni(\mathcal{K} \otimes \mathcal{K}' \otimes \mathcal{H})$$

is a unitary representation of $E(2)$ acting on $\mathcal{K} \otimes \mathcal{K}'$. By our theorem this representation is related to an element of $\hat{\mathcal{G}}_{\mathcal{K}\otimes\mathcal{K}'}$. This element is denoted by $\hat{g} \bigoplus \hat{g}'$. So we have

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$$\hat{g} \bigoplus \hat{g} = (N \otimes I_{\mathcal{K}'} \dot{+} I_{\mathcal{K}} \otimes \hat{N}', \; \hat{m} \otimes q^{2\hat{N}'} \dot{+} I_{\mathcal{K}} \otimes \hat{m}')$$

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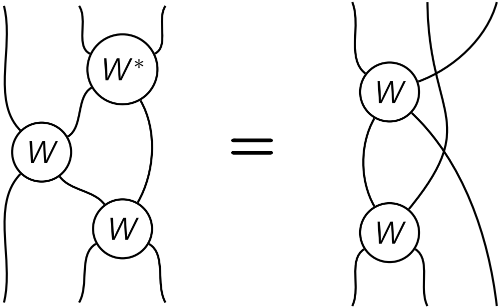
Multiplicative Unitaries

Let $\mathcal{H}$ be a Hilbert space and W be a unitary operator acting on $\mathcal{H} \otimes \mathcal{H}$. W is called a multiplicative unitary if

$$W_{23} W_{12} = W_{12} W_{13} W_{23}$$



Modular multiplicative unitaries

A unitary $W \in \mathcal{U}ni(\mathcal{H} \otimes \mathcal{H})$ is a modular multiplicative unitary if

- W is multiplicative $(W_{23}W_{12} = W_{12}W_{13}W_{23})$
- there exist positive, selfadjoint Q and $\hat{Q}$ on $\mathcal{H}$ such that

$$W\left(\hat{Q} \otimes Q\right)W^* = \hat{Q} \otimes Q$$

- we have

$$(x \otimes y|W|z \otimes u) = \left(\overline{z} \otimes Qy\middle|\widetilde{W}\middle|\overline{x} \otimes Q^{-1}u\right)$$

for a certain unitary $\widetilde{W} \in \mathcal{U}ni(\bar{\mathcal{H}} \otimes \mathcal{H})$.

W is manageable if $\hat{Q} = Q$.

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From multiplicative unitaries to quantum groups

- Take modular m.u. $W \in \mathcal{U}ni(\mathcal{H} \otimes \mathcal{H})$
- Let $A = \{\omega \otimes \mathrm{id})(W) : \omega \in B(\mathcal{H})_*\}^{\text{norm closure}} \subset B(\mathcal{H})$
- A is a C^*-algebra.
- For $a \in A$ we have $W(a \otimes I_{\mathcal{H}})W^* \in M(A \otimes A)$ and

$$A \ni a \longmapsto W(a \otimes I_{\mathcal{H}})W^*$$

defines a comultiplication $\Delta \in \mathrm{Mor}(A, A \otimes A)$.

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- We have $(\Delta \otimes \mathrm{id})\circ\Delta = (\mathrm{id} \otimes \Delta)\circ\Delta$
- We have

$$\left\{\Delta(a)(I \otimes b) : a, b \in A\right\}^{\mathrm{CLS}} = A \otimes A,$$
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- There is a closed antimultiplicative map

$$\kappa : (\omega \otimes \mathrm{id})(W) \longmapsto (\omega \otimes \mathrm{id})(W^*)$$

- Moreover $\kappa = R\circ\tau_{\frac{i}{2}}$ where R is an antiautomorphism of A and $(\tau_t)_{t\in\mathbb{R}}$ is a one parameter group of automorphisms of A: $\tau_t(a) = Q^{2it} a Q^{-2it}$

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Heisenberg commutation relations

Let $\hat{g} \in \hat{\mathcal{G}}_{\mathcal{H}}$ *and* $g \in \mathcal{G}_{\mathcal{H}}$ *(the same Hilbert space!). We say that* $(\hat{g}, g)$ *is a Heisenberg pair if*

$$U(\hat{g}, g')(g \otimes I_{\mathcal{K}})U(\hat{g}, g')^* = g \bigoplus g'$$

for any $g' \in \mathcal{G}_{\mathcal{K}}$. *For any Hilbert space* $\mathcal{H}$ *we set*

$$\mathfrak{H}_{\mathcal{H}} = \left\{ (\hat{g}, g) \in \hat{\mathcal{G}}_{\mathcal{H}} \times \mathcal{G}_{\mathcal{H}} : \begin{array}{c} (\hat{g}, g) \text{ is a} \\ \text{Heisenberg pair} \end{array} \right\}$$

Then $\mathfrak{H}$ is a closed complete operator domain.

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Heisenberg relations and multiplicative unitaries

Theorem

Let $\mathcal{H}$ be a Hilbert space and $(\hat{g}, g) \in \mathfrak{H}_{\mathcal{H}}$. Then

$$W = U(\hat{g}, g)$$

is a multiplicative unitary acting on $\mathcal{H} \otimes \mathcal{H}$ related to the quantum $E(2)$-group. Any multiplicative unitary related to the quantum $E(2)$-group is of this form.

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Heisenberg relations revealed

Theorem

Let $\hat{N}, \hat{m}, u, m$ be closed operators acting on a Hilbert space $\mathcal{H}$. Then $(\hat{N}, \hat{m}, u, m) \in \mathfrak{H}_{\mathcal{H}}$ if and only if the following conditions are satisfied:

- $(u, m) \in \mathcal{G}_{\mathcal{H}}$, $\ker m = \{0\}$,
- $u^* \hat{N} u = \hat{N} + I_{\mathcal{H}}$
- $\hat{N}$ *and* m *strongly commute.*
- $\hat{m} = m^{-1} u \dotplus \hat{r}$, *where* $\hat{r}$ *is a closed operator such that:*
    - $(\hat{N}, \hat{r}) \in \hat{\mathcal{G}}_{\mathcal{H}}$,
    - $u^* \hat{r} u = q^2 \hat{r}$,
    - $\hat{r}$ *and* m *strongly commute.*

The case $\hat{r} = 0$.

We say that the Heisenberg pair is of type I if $\hat{r} = 0$.

In this case we put $v = \mathrm{Phase}\, m$, $N = \log_q |m|$ and $\hat{v} = uv^{-2}$. Then

- $\hat{v}, v$ are unitary,
- $\hat{N}, N$ are selfadjoint with integer spectrum,
- $\hat{v}\hat{N}\hat{v}^* = \hat{N} + I$ and $vNv^* = N + I$,
- $(\hat{N}, \hat{v})$ strongly commutes with (N, v).

$(\hat{N}, \hat{v})$ and (N, v) are Heisenberg pair for the abelian group S^1. Hence quadruple $(\hat{N}, \hat{v}, N, v)$ is UNIQUE.

Theorem

The Heisenberg pair of type I is unique (up to the unitary equivalence and multiplicity).

The case $\hat{r} = 0$.

We say that the Heisenberg pair is of type I if $\hat{r} = 0$.

In this case we put $v = \mathrm{Phase}\, m$, $N = \log_q |m|$ and $\hat{v} = uv^{-2}$. Then

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- $\hat{v}$, v are unitary,
- $\hat{N}'$, N are selfadjoint with integer spectrum,
- $\hat{v}\hat{N}'\hat{v}^* = \hat{N}' + I$ and $vNv^* = N + I$,
- $\hat{r}' \in \mathcal{F}^*$, $\ker \hat{r}' = \{0\}$,
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Structure of Heisenberg pair of type II

Simple observation:

If $(\hat{g}, g) \in \mathfrak{H}_{\mathcal{H}}$ *and* $\hat{g}' \in \hat{\mathcal{G}}_{\mathcal{K}}$ *then*

$$(\hat{g}' \bigoplus \hat{g}, I_{\mathcal{K}} \otimes g) \in \mathfrak{H}_{\mathcal{K}\otimes\mathcal{H}}$$

Theorem

If $(\hat{g}, g)$ *is a Heisenberg pair of type I and* $\hat{g}' = (\hat{N}', \hat{m}') \in \hat{\mathcal{G}}_{\mathcal{K}}$ *with* $\ker \hat{m}' = \{0\}$ *then*

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is a Heisenberg pair of type II. Any Heisenberg pair of type II arises in this way.

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Multiplicative Unitaries

- MU of type I are modular and regular.
- MU of type II are manageable and non-regular.
- MU are labelled by pairs of nonnegative integers (k, l) with $k + l \geq 1$.