Chords, Norms, and q-Commutation Relations

Natasha Blitvic
MIT

ESI
April 19, 2011

CCR and ACR

Canonical commutation relations (CCR):

$$\ell_i^* \ell_j - \ell_j \ell_i^* = \delta_{ij} \qquad \text{Bosons}$$

Canonical anti-commutation relations (ACR):

$$\ell_i^* \ell_j + \ell_j \ell_i^* = \delta_{ij} \qquad \text{Fermions}$$

Interpolation: the q-commutation relations

$$\ell_i^* \ell_j - q \ell_j \ell_i^* = \delta_{ij} \quad \text{for } -1 \leq q \leq 1.$$

q-Fock Space

[Bożejko & Speicher]
For $\mathscr{H}$ a Hilbert space:

$$\mathcal{F}(\mathscr{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathscr{H}^{\otimes n} \qquad \text{(algebraically)}$$

For $q \in (-1, 1)$, an inner product on $\mathcal{F}(\mathscr{H})$:

$$(f_1 \oplus \ldots \oplus f_j, g_1 \oplus \ldots \oplus g_k)_q := \delta_{j,k} \sum_{\pi \in S_k} q^{\#\text{ inversions in }\pi} (f_1, g_{\pi,1}) \ldots (f_k, g_{\pi,k})$$

Then, $\mathcal{F}_q(\mathscr{H}) :=$ completion of $\mathcal{F}(\mathscr{H})$.

q-commuting "random variables"

For $f \in \mathscr{H}$, *creation operator* $\ell_q(f)$ on $\mathcal{F}_q(\mathscr{H})$:

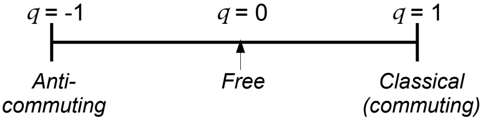
$$\ell_q(f)\, f_1 \otimes \ldots \otimes f_k = f \otimes f_1 \otimes \ldots \otimes f_k.$$

Adjoint = *anihilation operator* $\ell_q^*(f)$.

Facts: for $q \in (-1, 1)$,

1. $\ell_q(f), \ell_q^*(f)$ are bounded.
2. $\mathcal{B}(\mathcal{F}_q(\mathscr{H}))$ = von Neumann algebra generated by $\{\ell_q(h) \mid h \in \mathscr{H}\}$.

"q-deformed probability":



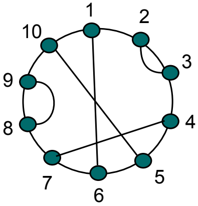
q-Semicircular and q-Circular

q-semicircular (q-Gaussian) s_q

[Bożejko & Speicher 1991]

$$s_q = \ell_1 + \ell_1^*$$

$$\varphi(s_q^{2n}) = \sum_{\pi \in \mathcal{P}(2n)} q^{\mathrm{cr}(\pi)}$$



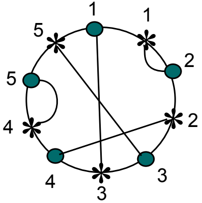
$$||s_q|| = \frac{2}{\sqrt{1-q}}$$

q-circular (q-complex Gaussian) c_q

[Mingo & Nica 2001, Kemp 2005]

$$c_q = \frac{\ell_1 + \ell_1^* + i(\ell_2 + \ell_2^*)}{\sqrt{2}}$$

$$\varphi((c_q c_q^*)^n) = \sum_{\pi \in \mathcal{P}(\circ *)^n} q^{\mathrm{cr}(\pi)}$$



$$||c_q|| = ?$$

$$\lim_{n\to\infty} ||s_q||_{2n} = ||s_q|| = \frac{2}{\sqrt{1-q}}$$

$$\lim_{n\to\infty} ||c_q||_{2n} = ||c_q|| = ?$$

Norms of the q-Semicircular

Theorem

Let $\gamma_1, \gamma_2, \dots$ be the sequence of $2n$-norms of the q-semicircular operator and consider the sequence of complex-valued functions $\tilde{\gamma}_1, \tilde{\gamma}_2, \dots$ defined on the unit ball $B_{\mathbb{C}} := \{z \in \mathbb{C}, |z| < 1\}$ as

$$\tilde{\gamma}_n(q) = \frac{1}{\sqrt{1-q}} \left(\sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \binom{2n}{n+k} \right)^{\frac{1}{2n}}.$$

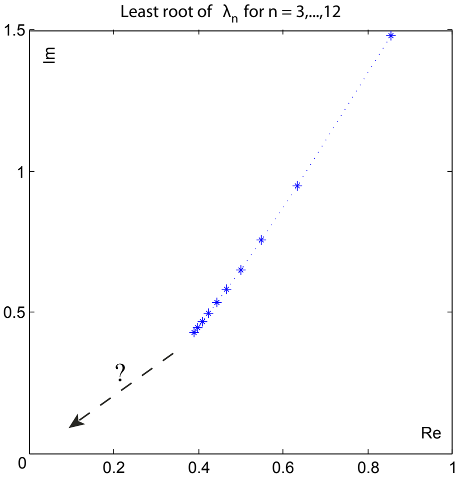
Then, for n large enough, $\tilde{\gamma}_n$ analytically extend γ_n on $B_{\mathbb{C}}$. Moreover,

$$\tilde{\gamma}_n(q) \to \frac{2}{\sqrt{1-q}}, \quad q \in B_{\mathbb{C}}$$

and the convergence is uniform on compact subsets of $B_{\mathbb{C}}$.

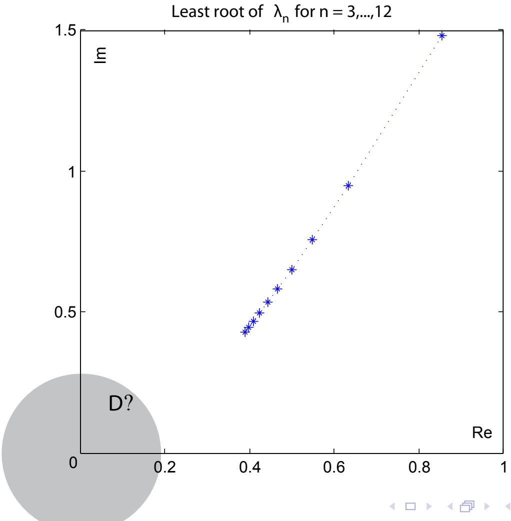
Norms of the q-Circular

Least-magnitude roots of the $2n$-norms of c_q for $n \leq 12$.



Norms of the q-Circular

Least-magnitude roots of the $2n$-norms of c_q for $n \leq 12$.



Norms of the q-Circular

Theorem

Let $\lambda_1, \lambda_2, \ldots$ be the sequence of $2n$-norms of the q-circular operator. Then, there exists no complex neighborhood of the origin on which λ_n have analytic continuations that converge uniformly on compact sets.

Norms of the q-Circular

Theorem

Let $\lambda_1, \lambda_2, \ldots$ be the sequence of $2n$-norms of the q-circular operator. Then, there exists no complex neighborhood of the origin on which λ_n have analytic continuations that converge uniformly on compact sets.

11

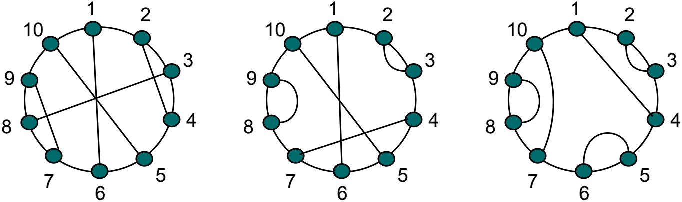
Combinatorics of s_q

$$\varphi(s_q^{2n}) = \sum_{\pi \in \mathcal{P}(2n)} q^{\mathrm{cr}(\pi)}$$

where

$$P(2n) = \text{pair partitions of } [2n],$$

$$\mathrm{cr}(\pi) = |\{\{a_1, b_1\}, \{a_2, b_2\} \in \pi \mid a_1 < a_2 < b_1 < b_2\}|.$$



A few moments:

$$\varphi(s_q^2) = 1$$

$$\varphi(s_q^4) = 2 + q$$

$$\varphi(s_q^6) = 5 + 6q + 3q^2 + q^3$$

$$\dots$$

A few moments:

$$\varphi(s_q^2) = 1$$

$$\varphi(s_q^4) = 2 + q$$

$$\varphi(s_q^6) = 5 + 6q + 3q^2 + q^3$$

$$\ldots$$

Let $t_{n,k} = \#$ pairings on $[2n]$ with k crossings.

$$\varphi(s_q^{2n}) = \sum_{\pi \in \mathcal{P}(2n)} q^{\mathrm{cr}(\pi)} = \sum_{k=0,1,\ldots} t_{n,k} q^k$$

$$t_{n,0} = \frac{1}{n+1}\binom{2n}{n}, \quad t_{n,1} = \frac{3}{2n+1}\binom{2n+1}{n-1}, \quad t_{n,k} = 0 \text{ for } k > \binom{n}{2}$$

Theorem (Touchard1952,Riordan1975)

For $n \in \mathbb{N}$,

$$\sum_{\pi \in \mathcal{P}(2n)} q^{cr(\pi)} = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \binom{2n}{n+k}.$$

Theorem (Touchard1952,Riordan1975)

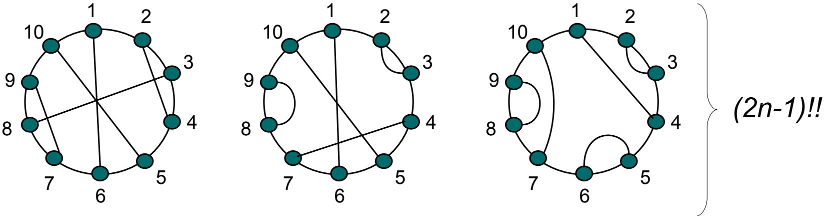
For $n \in \mathbb{N}$,

$$\sum_{\pi \in \mathcal{P}(2n)} q^{cr(\pi)} = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \binom{2n}{n+k}.$$

$\Longrightarrow$ moments of s_q, convergence of norms, etc.

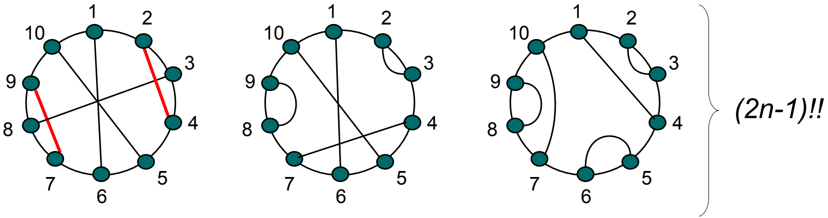
Combinatorics of c_q

$$\varphi((c_q c_q^*)^n) = \sum_{\pi \in \mathcal{P}(\circ *)^n} q^{\mathrm{cr}(\pi)}$$



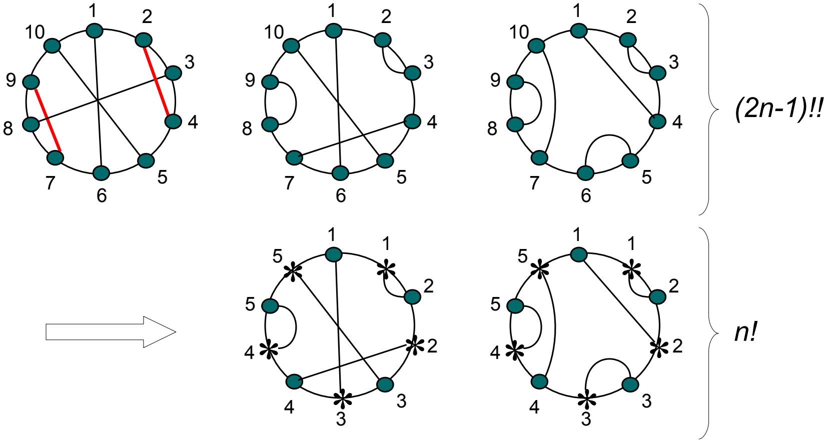
Combinatorics of c_q

$$\varphi((c_q c_q^*)^n) = \sum_{\pi \in \mathcal{P}(\circ *)^n} q^{\mathrm{cr}(\pi)}$$



Combinatorics of c_q

$$\varphi((c_q c_q^*)^n) = \sum_{\pi \in \mathcal{P}(\circ *)^n} q^{\mathrm{cr}(\pi)}$$

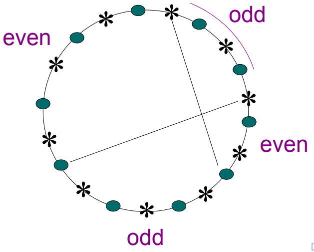


Combinatorics of c_q

Let $r_{n,k} = |\{\pi \in \mathcal{P}(\circ *)^n \text{ with } k \text{ crossings}\}|$.

$$\varphi((c_q c_q^*)^n) = \sum_{\pi \in \mathcal{P}(\circ *)^n} q^{\mathrm{cr}(\pi)} = \sum_{k=0}^{\binom{n}{2}} r_{n,k} q^k$$

$$r_{n,0} = \frac{1}{n+1}\binom{2n}{n}, \quad r_{n,k} = 0 \text{ for } k > \binom{n}{2}, \quad r_{n,1} = r_{n,2} = 0$$



Combinatorics of c_q

$$r_{n,3} = \binom{2n}{n-3}.$$

Combinatorics of c_q

$$r_{n,3} = \binom{2n}{n-3}.$$

$$r_{n,4} = 2\binom{2n}{n-4}.$$

Combinatorics of c_q

$$r_{n,3} = \binom{2n}{n-3}.$$

$$r_{n,4} = 2\binom{2n}{n-4}.$$

$$r_{n,5} = 2\binom{2n}{n-5}.$$

Combinatorics of c_q

$$r_{n,3} = \binom{2n}{n-3}.$$

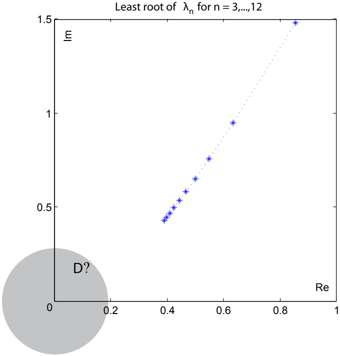
$$r_{n,4} = 2\binom{2n}{n-4}.$$

$$r_{n,5} = 2\binom{2n}{n-5}.$$

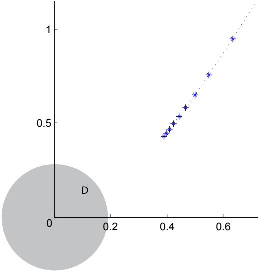
$$r_{n,6} = 2\binom{2n}{n-6} + 5\binom{2n}{n-5} + \frac{n+6}{2!}\binom{2n}{n-6}.$$

Back to the original question

$$\lambda_n = \varphi((c_q c_q^*)^n)^{\frac{1}{2n}} = \left(\sum_{k=0}^{\binom{n}{2}} r_{n,k} q^k \right)^{\frac{1}{2n}}$$



Back to the original question



Assume to the contrary that λ_n are analytic on $D \ni 0$ and converge uniformly on its compact subsets.

$$\implies \lim_{n\to\infty} \left(\frac{d^m}{dq^m} \lambda_n(q) \right) \Big|_{q=0} = \frac{d^m}{dq^m} \lambda(q) \Big|_{q=0}.$$

(Secretly hoping derivative $\to \infty$ for some m.)

Note:

$$\frac{d^m}{dq^m}\lambda_n(q)\Big|_{q=0} = \sum_{\substack{\ell_1,\ell_2,\ldots,\ell_m\in\{0,1,\ldots,m\}\\ \ell_1+2\ell_2+\ldots+m\ell_m=m}} \frac{\frac{1}{2n}!}{(\frac{1}{2n}-(\ell_1+\ldots+\ell_m))!}\frac{1}{\ell_1!\ldots\ell_m!} \times\left(\frac{r_{n,1}}{r_{n,0}}\right)^{\ell_1}\ldots\left(\frac{r_{n,m}}{r_{n,0}}\right)^{\ell_m}(r_{n,0})^{\frac{1}{2n}}$$

$\boxed{\text{Key 1:}}$ need $r_{n,k}$ only for $k \leq m$.

Diagram decomposition along $NC_{\text{even}}(2n)$

Observation: $\exists$ a unique decomposition

$$\text{pairing on } (\circ *)^n \quad \longleftrightarrow \quad \pi = \{V_1, \ldots, V_m\} \in NC_{\text{even}}(2n)$$
$$+ \text{ choice of connected pairing for each } V_i$$

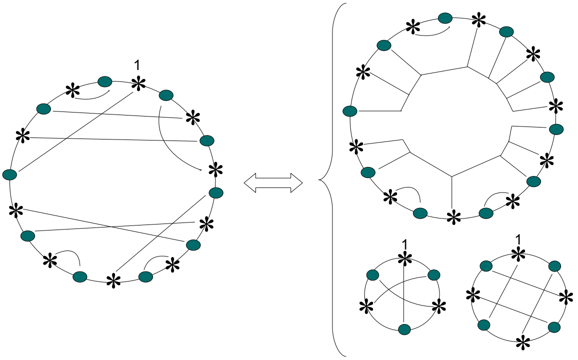
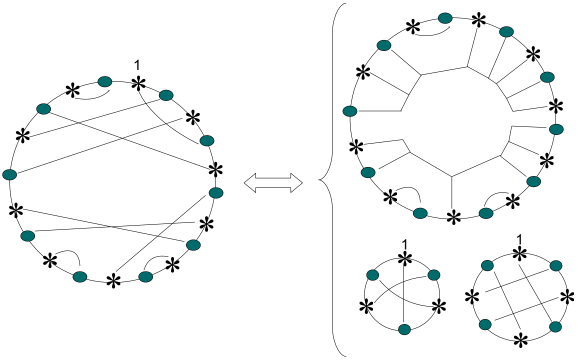


Diagram decomposition along $NC_{\text{even}}(2n)$

Observation: $\exists$ a unique decomposition

$$\text{pairing on } (\circ *)^n \quad \longleftrightarrow \quad \pi = \{V_1, \ldots, V_m\} \in NC_{\text{even}}(2n)$$
$$+ \text{ choice of connected pairing for each } V_i$$



Let $b_{n,k} = \#$ **connected** pairings on $(\circ *)^n$ with k crossings.

Let $b_{n,k} = \#$ **connected** pairings on $(\circ *)^n$ with k crossings.
Then,

$$r_{n,k} = \sum_{\substack{\ell=1,\dots,k \\ \beta=\{(n_1,k_1),\dots,(n_\ell,k_\ell)\} \\ n_1+\dots+n_\ell=n \\ k_1+\dots+k_\ell=k \\ n_i,k_i\in Z_+}} \frac{(2n)!}{\Phi_1(\beta)!\Phi_2(\beta)!\dots\Phi_n(\beta)!(2n+1-l)!} b_{n_1,k_1}\dots b_{n_\ell,k_\ell},$$

where $\Phi_i(\beta)$ counts the number of pairs β with the first coordinate equaling i, i.e. $\Phi_i(\beta) = \left|\{(\tilde{n},\tilde{k}) \in \beta \mid \tilde{n} = i\}\right|$.

Let $b_{n,k} = \#$ **connected** pairings on $(\circ *)^n$ with k crossings. Then,

$$r_{n,k} = \sum_{\substack{\ell=1,\ldots,k \\ \beta=\{(n_1,k_1),\ldots,(n_\ell,k_\ell)\} \\ n_1+\ldots+n_\ell=n \\ k_1+\ldots+k_\ell=k \\ n_i,k_i\in Z_+}} \frac{(2n)!}{\Phi_1(\beta)!\Phi_2(\beta)!\ldots\Phi_n(\beta)!(2n+1-l)!} b_{n_1,k_1}\ldots b_{n_\ell,k_\ell},$$

where $\Phi_i(\beta)$ counts the number of pairs β with the first coordinate equaling i, i.e. $\Phi_i(\beta) = \left|\{(\tilde{n},\tilde{k}) \in \beta \mid \tilde{n} = i\}\right|$.

But, $b_{n,k} = 0$ if $n > k$.

Let $b_{n,k} = \#$ **connected** pairings on $(\circ *)^n$ with k crossings.
Then,

$$r_{n,k} = \sum_{\substack{\ell=1,\ldots,k \\ \beta=\{(n_1,k_1),\ldots,(n_\ell,k_\ell)\} \\ n_1+\ldots+n_\ell=n \\ k_1+\ldots+k_\ell=k \\ n_i,k_i\in Z_+}} \frac{(2n)!}{\Phi_1(\beta)!\Phi_2(\beta)!\ldots\Phi_n(\beta)!(2n+1-l)!} b_{n_1,k_1}\ldots b_{n_\ell,k_\ell},$$

where $\Phi_i(\beta)$ counts the number of pairs β with the first coordinate equaling i, i.e. $\Phi_i(\beta) = \left|\{(\tilde{n},\tilde{k}) \in \beta \mid \tilde{n} = i\}\right|$.

But, $b_{n,k} = 0$ if $n > k$.

$\boxed{\text{Key 2:}}$ can derive $r_{n,k}$ from $b_{\ell,t}$ for $t, \ell \leq k$.

	n=1	2	3	4	5	6	7	8	9	10	11
k=0	1	0	0	0	0	0	0	0	0	0	0
k=3	0	0	1	0	0	0	0	0	0	0	0
4	0	0	0	2	0	0	0	0	0	0	0
5	0	0	0	0	2	0	0	0	0	0	0
6	0	0	0	0	5	2	0	0	0	0	0
7	0	0	0	0	5	24	2	0	0	0	0
8	0	0	0	0	0	18	56	2	0	0	0
9	0	0	0	0	0	4	70	176	2	0	0
10	0	0	0	0	1	12	98	328	576	2	0
11	0	0	0	0	0	12	105	408	1107	300	2

Table: $b_{n,k}$ for $0 \leq n, k \leq 11$

$\Longrightarrow \; r_{n,k} = \ldots$ for $k \leq 11$.

	n=1	2	3	4	5	6	7	8	9	10	11
k=0	1	0	0	0	0	0	0	0	0	0	0
k=3	0	0	1	0	0	0	0	0	0	0	0
4	0	0	0	2	0	0	0	0	0	0	0
5	0	0	0	0	2	0	0	0	0	0	0
6	0	0	0	0	5	2	0	0	0	0	0
7	0	0	0	0	5	24	2	0	0	0	0
8	0	0	0	0	0	18	56	2	0	0	0
9	0	0	0	0	0	4	70	176	2	0	0
10	0	0	0	0	1	12	98	328	576	2	0
11	0	0	0	0	0	12	105	408	1107	300	2

Table: $b_{n,k}$ for $0 \leq n, k \leq 11$

$\Longrightarrow$ $r_{n,k} = \ldots$ for $k \leq 11$.

$\Longrightarrow$ $\lim_{n\to\infty} \left(\frac{d^m}{dq^m} \lambda_n(q) \right) \Big|_{q=0} \in \mathbb{R}$ for $k < 11$.

$\lim_{n\to\infty} \left(\frac{d^{11}}{dq^{11}} \lambda_n(q) \right) \Big|_{q=0} = -\infty$. □

	n=1	2	3	4	5	6	7	8	9	10	11
k=0	1	0	0	0	0	0	0	0	0	0	0
k=3	0	0	1	0	0	0	0	0	0	0	0
4	0	0	0	2	0	0	0	0	0	0	0
5	0	0	0	0	2	0	0	0	0	0	0
6	0	0	0	0	5	2	0	0	0	0	0
7	0	0	0	0	5	24	2	0	0	0	0
8	0	0	0	0	0	18	56	2	0	0	0
9	0	0	0	0	0	4	70	176	2	0	0
10	0	0	0	0	1	12	98	328	576	2	0
11	0	0	0	0	0	12	105	408	1107	300	2

Table: $b_{n,k}$ for $0 \leq n, k \leq 11$

$$\Longrightarrow \quad r_{n,k} = \ldots \text{ for } k \leq 11.$$

$$\Longrightarrow \quad \lim_{n\to\infty} \left(\frac{d^m}{dq^m} \lambda_n(q) \right) \Big|_{q=0} \in \mathbb{R} \text{ for } k < 11.$$

$$\lim_{n\to\infty} \left(\frac{d^{11}}{dq^{11}} \lambda_n(q) \right) \Big|_{q=0} = -\infty. \qquad \square$$

A better way to do this. . .

$$\frac{d^m}{dq^m}\lambda_n(q)\Big|_{q=0} = \sum_{\substack{\ell_1,\ell_2,\ldots,\ell_m\in\{0,1,\ldots,m\}\\ \ell_1+2\ell_2+\ldots+m\ell_m=m}} \frac{\frac{1}{2n}!}{(\frac{1}{2n}-(\ell_1+\ldots+\ell_m))!}\frac{1}{\ell_1!\ldots\ell_m!} \times\left(\frac{r_{n,1}}{r_{n,0}}\right)^{\ell_1}\ldots\left(\frac{r_{n,m}}{r_{n,0}}\right)^{\ell_m}(r_{n,0})^{\frac{1}{2n}}$$

$$r_{n,k} = \sum_{\substack{\ell=1,\ldots,k\\ \beta=\{(n_1,k_1),\ldots,(n_\ell,k_\ell)\}\\ n_1+\ldots+n_\ell=n\\ k_1+\ldots+k_\ell=k\\ n_i,k_i\in Z_+}} \frac{(2n)!}{\Phi_1(\beta)!\Phi_2(\beta)!\ldots\Phi_n(\beta)!(2n+1-l)!} b_{n_1,k_1}\ldots b_{n_\ell,k_\ell}$$

$$\implies \frac{d^k}{dq^k}\lambda_n(q)\Big|_{q=0} = (r_{n,0})^{\frac{1}{2n}} \sum_{\substack{\ell=1,\dots,k \\ \beta=\{(n_1,k_1),\dots,(n_\ell,k_\ell)\} \\ n_1+n_2+\dots+n_\ell\le n \\ k_1+k_2+\dots+k_\ell=k \\ n_i,k_i\ge 3}} b_{n_1,k_1}\dots b_{n_\ell,k_\ell}\Upsilon(\beta)$$

$$\Longrightarrow \frac{d^k}{dq^k}\lambda_n(q)\Big|_{q=0} = (r_{n,0})^{\frac{1}{2n}} \sum_{\substack{\ell=1,\dots,k \\ \beta=\{(n_1,k_1),\dots,(n_\ell,k_\ell)\} \\ n_1+n_2+\dots+n_\ell\leq n \\ k_1+k_2+\dots+k_\ell=k \\ n_i,k_i\geq 3}} b_{n_1,k_1}\dots b_{n_\ell,k_\ell}\Upsilon(\beta)$$

Fact 1: Generally, $\Upsilon(\beta) \sim c_\beta\, n^{\ell-1}$.

$$\implies \frac{d^k}{dq^k}\lambda_n(q)\Big|_{q=0} = (r_{n,0})^{\frac{1}{2n}} \sum_{\substack{\ell=1,\ldots,k \\ \beta=\{(n_1,k_1),\ldots,(n_\ell,k_\ell)\} \\ n_1+n_2+\ldots+n_\ell\leq n \\ k_1+k_2+\ldots+k_\ell=k \\ n_i,k_i\geq 3}} b_{n_1,k_1}\ldots b_{n_\ell,k_\ell}\Upsilon(\beta)$$

Fact 1: Generally, $\Upsilon(\beta) \sim c_\beta\, n^{\ell-1}$.

Fact 2: If $\beta = \{(n_1, k_1), \ldots, (n_\ell, k_\ell)\}$ and $n_i = n_j \iff k_i = k_j$, then $\Upsilon(\beta) = \Theta(1)$.

Why 11?

For $k = 9$,

$$\beta_1 = \{(3,3),(3,3),(3,3)\}, \beta_2 = \{(4,4),(5,5)\},$$
$$\beta_3 = \{(3,3),(6,6)\}, \beta_4 = \{(3,3),(5,6)\}.$$

For $k = 10$, ...

For $k = 11$,

$$\beta_1 = \{(3,3),(3,3),(5,5)\}, \beta_2 = \{(3,3),(4,4),(4,4)\},$$
$$\beta_3 = \{(3,3),(6,8)\},\ \beta_4 = \{(3,3),(7,8)\},\ \beta_5 = \{(3,3),(8,8)\},$$
$$\beta_6 = \{(4,4),(5,7)\},\ \beta_7 = \{(4,4),(6,7)\},\ \beta_8 = \{(4,4),(7,7)\},$$
$$\beta_9 = \{(5,5),(6,6)\},\ \beta_{10} = \{(5,5),(5,6)\}.$$

Why 11?

For $k = 9$,

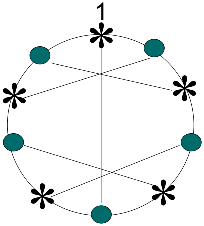
$$\beta_1 = \{(3,3),(3,3),(3,3)\}, \beta_2 = \{(4,4),(5,5)\},$$
$$\beta_3 = \{(3,3),(6,6)\}, \beta_4 = \{(3,3),(5,6)\}.$$

For $k = 10$, ...

For $k = 11$,

$$\beta_1 = \{(3,3),(3,3),(5,5)\}, \beta_2 = \{(3,3),(4,4),(4,4)\},$$
$$\beta_3 = \{(3,3),(6,8)\},\ \beta_4 = \{(3,3),(7,8)\},\ \beta_5 = \{(3,3),(8,8)\},$$
$$\beta_6 = \{(4,4),(5,7)\},\ \beta_7 = \{(4,4),(6,7)\},\ \beta_8 = \{(4,4),(7,7)\},$$
$$\beta_9 = \{(5,5),(6,6)\},\ \beta_{10} = \{(5,5),(5,6)\}.$$

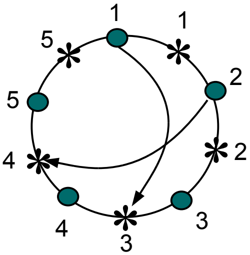
$n = 5, k = 6$



$\square$

Crossings are cool

Directed Crossings of Corteel (Adv. Appl. Math. 38 (2007))



1. q-commutation relation in the PASEP matrix ansatz: $DE - qED = D + E$. (Corteel & Williams, Adv. Appl. Math. 39 (2007))
2. Specialization of *staircase tableaux* of Corteel & Williams ↔ moments of Askey-Wilson OPS and stationary distribution of the ASEP. (Corteel & Williams, PNAS, March, 2010).

DANKE SCHÖN