Free subordination property and deformed matricial models

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Notation:
For any $N \times N$ hermitian matrix X,
$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_N(X)$ eigenvalues of X.

$$\mu_X := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(X)}$$

Definition

W_N is a $N \times N$ Wigner Hermitian matrix associated with a distribution μ of variance σ^2 and mean zero:

$(W_N)_{ii}$, $\sqrt{2}\Re e((W_N)_{ij})_{i<j}$, $\sqrt{2}\Im m((W_N)_{ij})_{i<j}$ are i.i.d, with distribution μ.

If $\mu = \mathcal{N}(0, \sigma^2)$, $W_N =: W_N^G$ is a $G.U.E$-matrix.

Theorem

Convergence of the spectral measure: *Wigner (50')*

$$\mu_{\frac{W_N}{\sqrt{N}}} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i\left(\frac{W_N}{\sqrt{N}}\right)} \to \mu_\sigma \quad a.s \text{ when } \quad N \to +\infty$$

$$\frac{d\mu_\sigma}{dx}(x) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}\,1_{[-2\sigma,2\sigma]}(x)$$

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Theorem

Convergence of the extremal eigenvalues *(Bai-Yin 1988):*
If $\int x^4 d\mu(x) < +\infty$, *then*

$$\lambda_1(\frac{W_N}{\sqrt{N}}) \to 2\sigma \text{ and } \lambda_N(\frac{W_N}{\sqrt{N}}) \to -2\sigma \text{ a.s when } N \to +\infty.$$

$$M_N = \frac{1}{\sqrt{N}} W_N + A_N$$

- W_N is a $N \times N$ Wigner Hermitian matrix associated with a distribution μ of variance σ^2 and mean zero which is symmetric and satisfies a Poincaré inequality.
- A_N is a deterministic Hermitian matrix.

$$\mu_{A_N} \to \nu \quad \text{weakly }, \nu \text{ compactly supported.}$$

The eigenvalues of A_N:

- $N - r$ (r fixed) eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N\to\infty} 0$$

- a **finite** number J of **fixed** (independent of N) eigenvalues **(spikes)** $\theta_1 > \ldots > \theta_J$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

μ satisfies a **Poincaré inequality**: there exists a positive constant C such that for any $\mathcal{C}^1$ function $f : \mathbb{R} \to \mathbb{C}$ such that f and f' are in $L^2(\mu)$,

$$\mathbb{E}_\mu(|f - \mathbb{E}_\mu(f)|^2) \leq C\mathbb{E}_\mu(|f'|^2).$$

Poincaré inequality is just a technical condition.

Anderson-Guionnet-Zeitouni, Mingo-Speicher $\Longrightarrow \frac{W_N}{\sqrt{N}}$ and A_N are asymptotically free almost surely.

$$\Longrightarrow \mu_{M_N} \xrightarrow{w} \mu_\sigma \boxplus \nu \text{ a.s.}$$

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Actually free probability will also shed light on the asymptotic behaviour of eigenvalues through free subordination property

For a probability measure τ on $\mathbb{R}$, $z \in \mathbb{C} \backslash \mathbb{R}$, $g_\tau(z) = \int_{\mathbb{R}} \frac{d\tau(x)}{z-x}$.

Theorem (D.Voiculescu, P. Biane)

Let μ and ν be two probability measures on $\mathbb{R}$, there exists a unique analytic map $F : \mathbb{C}^+ \to \mathbb{C}^+$ such that

$$\forall z \in \mathbb{C}^+, g_{\mu \boxplus \nu}(z) = g_\nu(F(z)),$$

$$\forall z \in \mathbb{C}^+, \Im F(z) \geq \Im z$$

and

$$\lim_{y \uparrow +\infty} \frac{F(iy)}{iy} = 1.$$

F is called the subordination map of $\mu \boxplus \nu$ with respect to ν.

μ_σ:the centered semi-circular distribution with variance σ^2;
ν: a probability measure on $\mathbb{R}$. Study of $\nu \boxplus \mu_\sigma$ by P.Biane 1997.

$$\forall z \in \mathbb{C}^+, \quad g_{\nu\boxplus\mu_\sigma}(z) = g_\nu(F_{\sigma,\nu}(z)); \quad F_{\sigma,\nu}(z) = z - \sigma^2 g_{\nu\boxplus\mu_\sigma}(z)$$

Theorem (P.Biane 1997)

$$F_{\sigma,\nu} : \begin{array}{l} \mathbb{C}^+ \to \{u + iv \in \mathbb{C}^+, v > v_{\sigma,\nu}(u)\} := \Omega_{\sigma,\nu} \\ z \mapsto z - \sigma^2 g_{\nu\boxplus\mu_\sigma}(z) \end{array}$$

$$v_{\sigma,\nu}(u) = \inf\left\{ v \geq 0, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2 + v^2} \leq \frac{1}{\sigma^2} \right\}.$$

$$H_{\sigma,\nu} : z \mapsto z + \sigma^2 g_\nu(z)$$

is a homeomorphism from $\overline{\Omega_{\sigma,\nu}}$ *to* $\mathbb{C}^+ \cup \mathbb{R}$ *with inverse* $F_{\sigma,\nu}$.

$$U_{\sigma,\nu} := \{u \in \mathbb{R}, v_{\sigma,\nu}(u) > 0\} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$$

$${}^c\text{support } \nu \boxplus \mu_\sigma \quad \underset{H_{\sigma,\nu}}{\overset{F_{\sigma,\nu}}{\rightleftarrows}} \quad {}^c\overline{U_{\sigma,\nu}} =: \Theta_{\sigma,\nu}; \qquad H_{\sigma,\nu} : z \mapsto z + \sigma^2 g_\nu(z).$$

Theorem (Characterization of the complementary of the support)

$$x \in^c \text{supp}(\mu_\sigma \boxplus \nu) \Leftrightarrow \exists u \in \Theta_{\sigma,\nu} \text{ such that } x = H_{\sigma,\nu}(u).$$

$$\Theta_{\sigma,\nu} = \left\{ u \in^c supp(\nu), \int \frac{1}{(u-t)^2} d\nu(t) < \frac{1}{\sigma^2} \right\}.$$

$$U_{\sigma,\nu} := \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$$

- Since ν is compactly supported, there exist

$s_m < t_m < \ldots < s_1 < t_1$ such that $\overline{U_{\sigma,\nu}} = \bigcup_{l=m}^{1} [s_l, t_l]$ and then

$\Theta_{\sigma,\nu} =^c \overline{U_{\sigma,\nu}} =]-\infty; s_m[\bigcup_{l=m}^{2}]t_l, s_{l-1}[\bigcup]t_1; +\infty[.$

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• The inverse of the subordination function $F_{\sigma,\nu}$,
$H_{\sigma,\nu}: \; z \mapsto z + \sigma^2 g_\nu(z)$ is globally increasing on $\overline{\Theta_{\sigma,\nu}}$.

$$\Longrightarrow \text{support } \nu \boxplus \mu_\sigma = \bigcup_{l=m}^{1} [H_{\sigma,\nu}(s_l), H_{\sigma,\nu}(t_l)]$$

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• Each connected component of $\overline{U_{\sigma,\nu}}$ contains at least a connected component of $\mathrm{supp}(\nu)$.

Example:

$$\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1},\ U_{\sigma,\nu} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\},$$

$$f : u \mapsto \int \frac{d\nu(x)}{(u-x)^2} = \frac{1}{2}\frac{1}{(u-1)^2} + \frac{1}{2}\frac{1}{(u+1)^2}.$$

$$\begin{array}{c|ccccccccc|}
u & -\infty & & -1 & & 0 & & 1 & & +\infty \\
\hline
 & & & +\infty & & & & +\infty & & \\
f(u) & & \nearrow & \| & \searrow & 1 & \nearrow & \| & \searrow & \\
 & 0 & & \| & & & & \| & & 0
\end{array}$$

Assume $\sigma > 1$. $f : u \mapsto \int \frac{d\nu(x)}{(u-x)^2} = \frac{1}{2}\frac{1}{(u-1)^2} + \frac{1}{2}\frac{1}{(u+1)^2}$.

u	$-\infty$		s_1		-1		0		1		t_1		$+\infty$
$f(u)$	0	$\nearrow$	$1/\sigma^2$	$\nearrow$	$+\infty \;\Vert\; +\infty$	$\searrow$	1	$\nearrow$	$+\infty \;\Vert\; +\infty$	$\searrow$	$1/\sigma^2$	$\searrow$	0

$$\overline{U_{\sigma,\nu}} = [s_1, t_1], \quad \mathrm{supp}\mu_\sigma \boxplus \nu = [H_{\sigma,\nu}(s_1), H_{\sigma,\nu}(t_1)]$$

$$H_{\sigma,\nu}(z) = z + \sigma^2 g_\nu(z) = z + \frac{\sigma^2}{2}\frac{1}{(z-1)} + \frac{\sigma^2}{2}\frac{1}{(z+1)}$$

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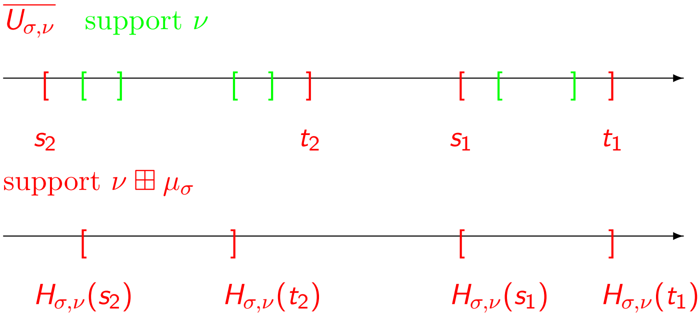
$\Longrightarrow$ It seems that for large N, the θ_i's in $\Theta_{\sigma,\nu}$ generate eigenvalues of M_N outside the support of the limiting spectral measure, in a neighborhood of the $\rho_{\theta_i} = H_{\sigma,\nu}(\theta_i)$...

$n_{i-1}+1, \ldots, n_{i-1}+k_i$: the descending ranks of θ_i among the eigenvalues of A_N.

Results (Capitaine-Donati-Martin-Féral-Février):

Complete description of the convergence of
$\lambda_{n_{i-1}+1}(M_N), \ldots, \lambda_{n_{i-1}+k_i}(M_N)$
depending on the location of θ_i with respect to $\overline{U_{\sigma,\nu}}$ and the connected components of $\mathrm{support}\ \nu$.


$\overline{U_{\sigma,\nu}}$
support ν
s_2
t_2
s_1
t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$
$H_{\sigma,\nu}(t_2)$
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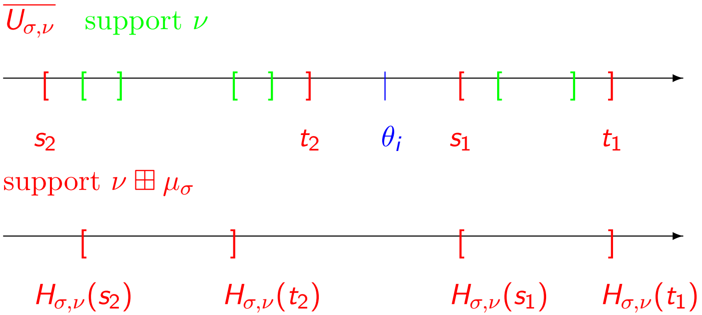
$\overline{U_{\sigma,\nu}}$ support ν
s_2
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θ_i
s_1
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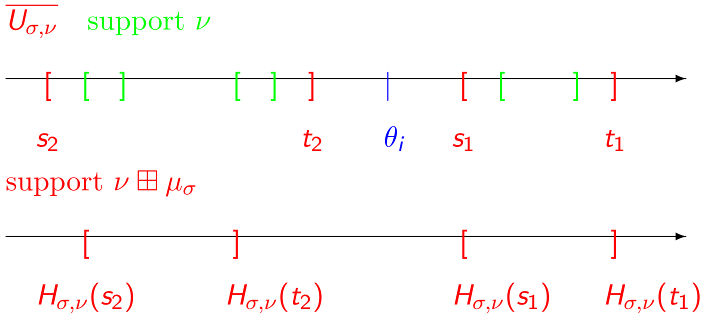

Theorem (Capitaine,Donati-Martin,Féral,Février)

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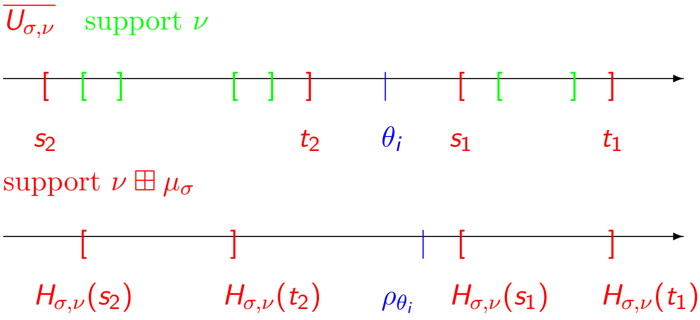
If $\theta_i \in \Theta_{\sigma,\nu} = {}^c\overline{U_{\sigma,\nu}}$, *the* k_i *eigenvalues* $(\lambda_{n_{i-1}+j}(M_N), 1 \le j \le k_i)$ *converge almost surely outside the support of* $\nu \boxplus \mu_\sigma$ *towards* $\rho_{\theta_i} = H_{\sigma,\nu}(\theta_i) = \theta_i + \sigma^2 g_\nu(\theta_i)$.



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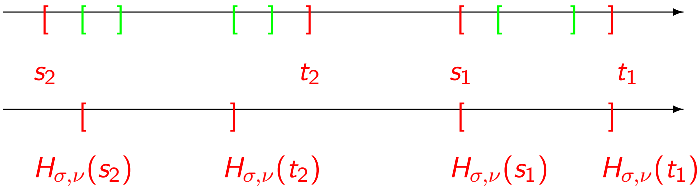
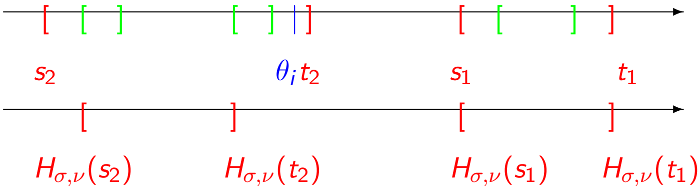
Results




s_2
θ_i t_2
s_1
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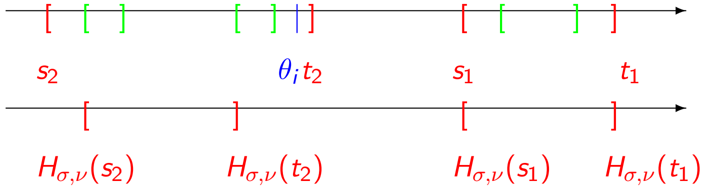

Theorem (Capitaine,Donati-Martin,Féral,Février 2010)

$$U_{\sigma,\nu} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}, \quad \overline{U_{\sigma,\nu}} = \bigcup_{l=m}^{1} [s_l, t_l]$$

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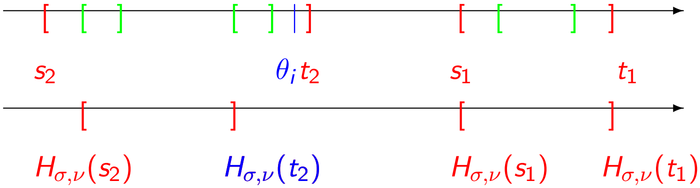
If $\theta_i \in \overline{U_{\sigma,\nu}}$ $\theta_i \in [s_{l_i}, t_{l_i}]$, θ_i *is on the right (resp. the left) of any connected component of* $\mathrm{supp}(\nu)$ *which is included in* $[s_{l_i}, t_{l_i}]$ *then the* k_i *eigenvalues* $(\lambda_{n_{i-1}+j}(M_N), 1 \le j \le k_i)$ *converge almost surely to* $H_{\sigma,\nu}(t_{l_i})$ *(resp.* $H_{\sigma,\nu}(s_{l_i})$*) which is a boundary point of the support of* $\nu \boxplus \mu_\sigma$.



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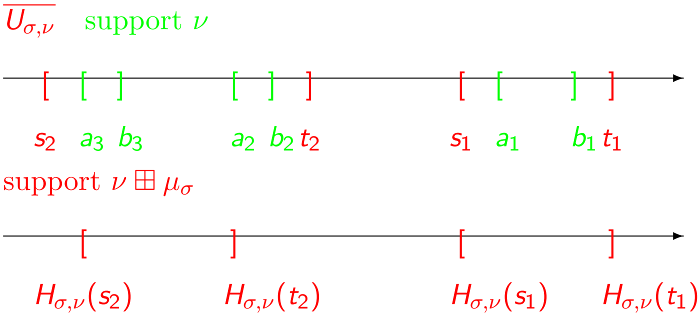
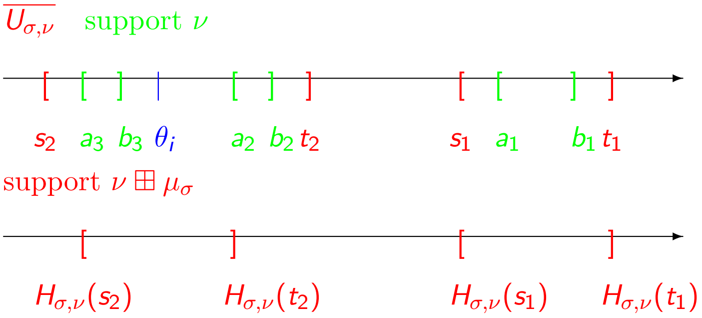
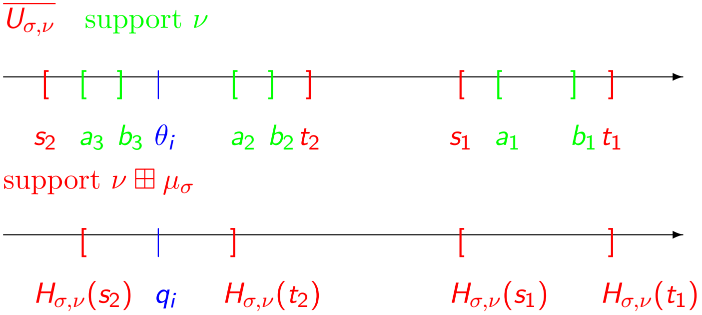

$\overline{U_{\sigma,\nu}}$ support ν
s_2 a_3 b_3 a_2 b_2 t_2 s_1 a_1 b_1 t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$ $H_{\sigma,\nu}(t_2)$ $H_{\sigma,\nu}(s_1)$ $H_{\sigma,\nu}(t_1)$

Results


$\overline{U_{\sigma,\nu}}$ support ν
s_2 a_3 b_3 θ_i a_2 b_2 t_2 s_1 a_1 b_1 t_1
support $\nu \boxplus \mu_\sigma$
$H_{\sigma,\nu}(s_2)$ $H_{\sigma,\nu}(t_2)$ $H_{\sigma,\nu}(s_1)$ $H_{\sigma,\nu}(t_1)$



Theorem

If θ_i is between two connected components of $\operatorname{supp}(\nu)$ which are included in $[s_{l_i}, t_{l_i}]$ then the k_i eigenvalues $(\lambda_{n_{i-1}+j}(M_N),$ $1 \leq j \leq k_i)$ converge almost surely to q_i defined by
$\nu \boxplus \mu_\sigma(]-\infty, q_i]) = \nu(]-\infty, \theta_i])$.

Example:

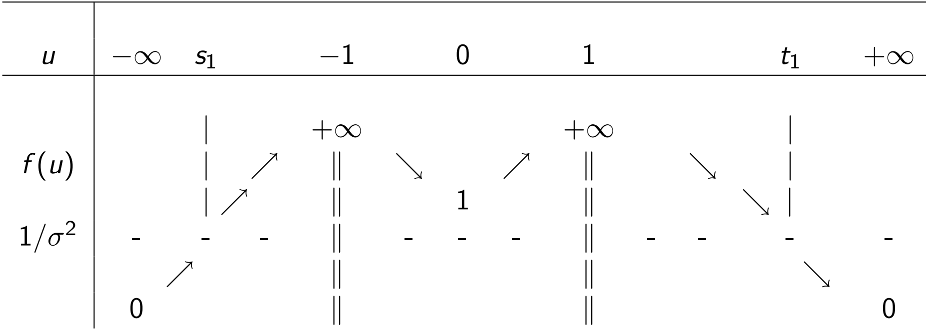
$$M_N = \frac{1}{\sqrt{N}} W_N + A_N, \quad A_N = \mathrm{diag}(\underbrace{-1,\dots,-1}_{\frac{N}{2}}, \theta, \underbrace{1,\dots,1}_{\frac{N}{2}-1}),$$

$\theta \notin \{-1;1\}$. $\nu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, $U_{\sigma,\nu} = \left\{ u \in \mathbb{R}, \int_{\mathbb{R}} \frac{d\nu(x)}{(u-x)^2} > \frac{1}{\sigma^2} \right\}$,

$f : u \mapsto \int \frac{d\nu(x)}{(u-x)^2}$.

u	$-\infty$		-1		0		1		$+\infty$
$f(u)$	0	$\nearrow$	$+\infty$ $\Vert$	$\searrow$	1	$\nearrow$	$+\infty$ $\Vert$	$\searrow$	0

Assume $\sigma > 1$. $\overline{U_{\sigma,\nu}} = [s_1, t_1]$, $\mathrm{supp}\mu_\sigma \boxplus \nu = [H_{\sigma,\nu}(s_1), H_{\sigma,\nu}(t_1)]$



If $\theta > t_1$, a.s $\lambda_1(M_N) \to \rho_\theta = \theta + \frac{1}{2}\sigma^2 \frac{1}{\theta - 1} + \frac{1}{2}\sigma^2 \frac{1}{\theta + 1} > H_{\sigma,\nu}(t_1)$.

Else, $\lambda_1(M_N) \to H_{\sigma,\nu}(t_1)$.

If $\theta < s_1$, a.s $\lambda_N(M_N) \to \rho_\theta = \theta + \frac{1}{2}\sigma^2 \frac{1}{\theta - 1} + \frac{1}{2}\sigma^2 \frac{1}{\theta + 1} < H_{\sigma,\nu}(s_1)$.

Else, $\lambda_N(M_N) \to H_{\sigma,\nu}(s_1)$.

If $-1 < \theta < 1$, a.s $\lambda_{\frac{N}{2}}(M_N) \to m$

$$m : \text{ median of } \nu \boxplus \mu_\sigma$$

The asymptotic behaviour of the eigenvalues of a deformed Wigner matrix comes from two phenomena:

- Inclusion of the spectrum of M_N in a ϵ-neighborhood of the support of $\mu_{A_N} \boxplus \mu_\sigma$ for all large N almost surely
- Exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, involving the subordination function $F_{\sigma,\nu}$.

For any $\epsilon > 0$,

Theorem

Almost surely, for all large N

$$Spect(\frac{1}{\sqrt{N}} W_N + A_N) \subset \epsilon\text{-neighborhood of support}(\mu_{A_N} \boxplus \mu_\sigma)$$

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$$+$$

for all large N,

$$support\,(\mu_{A_N} \boxplus \mu_\sigma) \subset \epsilon\text{-neighborhood of support } (\nu \boxplus \mu_\sigma) \bigcup_{i, |\theta_i \in^c \overline{U_{\sigma,\nu}}} \{\rho_{\theta_i}\}.$$

$$\Downarrow$$

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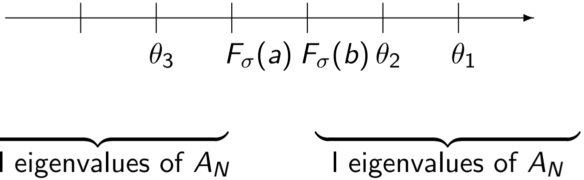
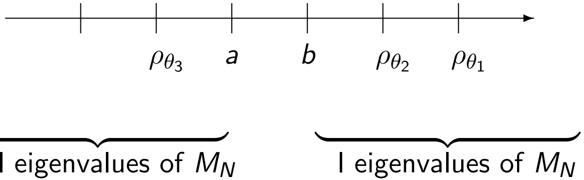
Exact separation phenomenon

$[a, b]$ gap in $\mathrm{Spect}(M_N) \longleftrightarrow [F_{\sigma,\nu}(a), F_{\sigma,\nu}(b)]$ gap in $\mathrm{Spect}(A_N)$



Then, almost surely, for all large N,



Theorem (M. Capitaine 2011)

θ_j in $\Theta_{\sigma,\nu}$; $n_{j-1}+1,\ldots,n_{j-1}+k_j$ the descending ranks of θ_j among the eigenvalues of A_N. $\xi_1(j),\ldots,\xi_{k_j}(j)$: an orthonormal system of eigenvectors associated to $(\lambda_{n_{j-1}+q}(M_N),\ 1\le q\le k_j)$. When N goes to infinity, for any $q\in\{1,\ldots,k_j\}$,

(i) *the square of the norm of the orthogonal projection of $\xi_q(j)$ onto the vector space Ker $(\theta_j I_N - A_N)$ converges a.s towards*

$$H'_{\sigma,\nu}(\theta_j) = 1 - \sigma^2 \int \frac{1}{(\theta_j - x)^2}\, d\nu(x),$$

(ii) *for any spiked eigenvalue θ_l of A_N such that $\theta_l \neq \theta_j$, the norm of the orthogonal projection of $\xi_q(j)$ onto Ker $(\theta_l I_N - A_N)$ converges almost surely towards zero when N goes to infinity.*

Definition

Wishart matrix associated to μ

$$X_N = \frac{1}{p} B_N B_N^*$$

B_N is a $N \times p(N)$ matrix,

$$(B_N)_{u,v} = Z_{u,v} + iY_{u,v}$$

$Z_{u,v}$, $Y_{u,v}$, $u = 1, \ldots, N$, $v = 1, \ldots, p(N)$ are i.i.d, with distribution μ with variance $\frac{1}{2}$ and mean zero.

If $\mu = \mathcal{N}(0, \frac{1}{2})$, X_N is a $L.U.E$ matrix.

Convergence of the spectral measure:

Theorem

Marchenko-Pastur *(1967):*

If $c_N := \frac{N}{p} \to c > 0$ *when* $N \to \infty$,

$$\mu_{\frac{B_N B_N^*}{p}} \to \mu_c \quad a.s \text{ when } \quad N \to +\infty$$

$$\frac{d\mu_c}{dx}(x) = \frac{1}{2\pi c x}\sqrt{(b-x)(x-a)}\,1_{[a,b]}(x)$$

$a = (1-\sqrt{c})^2$, $b = (1+\sqrt{c})^2$,

and $\mu_c(0) = 1 - \frac{1}{c}$ *if* $c > 1$.

Convergence of the largest eigenvalue

Theorem

If $\int x^4 d\mu(x) < +\infty$,
(Geman 1980) (Bai-Yin-Krishnaiah 1988) (Bai-Silverstein-Yin 1988)

$$\lambda_1\left(\frac{B_N B_N^*}{p(N)}\right) \to (1+\sqrt{c})^2 \ \textit{a.s when } N \to +\infty.$$

$$\lambda_{min(N,p)}\left(\frac{B_N B_N^*}{p(N)}\right) \to (1-\sqrt{c})^2 \ \textit{a.s when } N \to +\infty.$$

$$M_N = \frac{1}{p} A_N^{\frac{1}{2}} B_N B_N^* A_N^{\frac{1}{2}}$$

A_N : $N \times N$ nonnegative definite deterministic matrix.

$$\mu_{A_N} \to \nu \quad \text{weakly }, \nu \text{ compactly supported.}$$

The eigenvalues of A_N:

- $N - r$ (r fixed) eigenvalues $\beta_i(N)$ such that

$$\max_{i=1}^{N-r} \mathrm{dist}(\beta_i(N), \mathrm{supp}(\nu)) \to_{N\to\infty} 0$$

- a **finite** number J of **fixed** (independent of N) eigenvalues **(spikes)** $\theta_1 > \ldots > \theta_J > 0$, $\forall i = 1, \ldots, J$, $\theta_i \notin \mathrm{supp}(\nu)$, each θ_j having a fixed multiplicity k_j, $\sum_j k_j = r$.

Convergence of the spectral measure:

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Extremal eigenvalues:

Theorem (R. Rao, J. Silverstein and Z.D Bai, J. Yao)

$\Theta_{c,\nu} = \left\{ u \in \mathbb{R} \setminus \{\mathrm{supp}(\nu) \cup 0\}, \int_{\mathbb{R}} \frac{x^2}{(u-x)^2} d\nu(x) < \frac{1}{c} \right\}$

$n_{i-1}+1, \ldots, n_{i-1}+k_i$: *the descending ranks of* θ_i *among the eigenvalues of* A_N.

If $\theta_i \in \Theta_{c,\nu}$, *the* k_i *eigenvalues* $(\lambda_{n_{i-1}+j}(M_N), 1 \leq j \leq k_i)$ *converge almost surely outside the support of* $\nu \boxtimes \mu_c$ *towards*

$\rho_{\theta_i} = \theta_i + c\theta_i \int_{\mathbb{R}} \frac{x}{(\theta_i - x)} d\nu(x)$.

Remark: when A_N is a finite rank perturbation of the identity matrix

$$A_N = \mathrm{diag}\left(\underbrace{1,\ldots,1}_{\mathrm{N-r}},\underbrace{\theta_1,\ldots,\theta_1}_{\mathrm{k_1}},\underbrace{\theta_2,\ldots,\theta_2}_{\mathrm{k_2}},\ldots,\underbrace{\theta_J,\ldots,\theta_J}_{\mathrm{k_J}}\right)$$

$\theta_i \neq 1$, $\nu = \delta_1$ (and therefore $\mu_c \boxtimes \nu = \mu_c$). We recover the BBP transition (pionneering work of Baik-Ben Arous-Péché (2005) in the Gaussian case):

- If $\theta_1 > 1+\sqrt{c}$, a.s when $N \to +\infty$

$$\lambda_1\left(\frac{1}{p}A_N^{\frac{1}{2}}B_N B_N^* A_N^{\frac{1}{2}}\right) \to \theta_1\left(1+\frac{c}{\theta_1 - 1}\right) > (1+\sqrt{c})^2$$

- If $\theta_1 \leq 1+\sqrt{c}$, a.s when $N \to +\infty$

$$\lambda_1\left(\frac{1}{p}A_N^{\frac{1}{2}}B_N B_N^* A_N^{\frac{1}{2}}\right) \to (1+\sqrt{c})^2$$

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$$\Psi_\tau(z) = \int \frac{tz}{1-tz} d\tau(t) = \frac{1}{z} g_\tau\left(\frac{1}{z}\right) - 1,$$

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Theorem (Biane; Belinschi-Bercovici)

Let $\tau \neq \delta_0$ and $\nu \neq \delta_0$ be two probability measures on $[0; +\infty[$. There exists a unique analytic map $F^{(m)}_{\tau,\nu}$ defined on $\mathbb{C} \setminus [0; +\infty[$ such that

$$\forall\, z \in \mathbb{C} \setminus [0; +\infty[,\ \Psi_{\nu \boxtimes \tau}(z) = \Psi_\nu(F^{(m)}_{\tau,\nu}(z))$$

and $\forall\, z \in \mathbb{C}^+$,

$$F^{(m)}_{\tau,\nu}(z) \in \mathbb{C}^+,\ F^{(m)}_{\tau,\nu}(\overline{z}) = \overline{F^{(m)}_{\tau,\nu}(z)},\ \arg(F^{(m)}_{\tau,\nu}(z)) \geq \arg(z).$$

When τ is the Marchenko-Pastur distribution μ_c,

$$F^{(m)}_{\mu_c,\nu}(z) = z - cz + cg_{\mu_c \boxtimes \nu}(\frac{1}{z}).$$

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The limiting values ρ_{θ_i} of the eigenvalues that separate from the bulk can be expressed as

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- Exact separation phenomenon between the spectrum of M_N and the spectrum of A_N, involving the function $z \mapsto \frac{1}{F^{(m)}_{\mu_c,\nu}(\frac{1}{z})}$.

Theorem (Capitaine 2011)

(if μ satisfies a Poincaré inequality) For each spiked eigenvalue θ_j, we denote by $n_{j-1}+1,\dots,n_{j-1}+k_j$ the descending ranks of θ_j among the eigenvalues of A_N. Let $\xi_1(j),\dots,\xi_{k_j}(j)$ be an orthonormal system of eigenvectors associated to $(\lambda_{n_{j-1}+q}(M_N),\ 1\le q\le k_j)$. Then when θ_j is in $\Theta_{c,\nu}$, when N goes to infinity, for any $q\in\{1,\dots,k_j\}$,

(i) *the square of the norm of the orthogonal projection of $\xi_q(j)$ onto the vector space Ker $(\theta_j I_N - A_N)$ converges a.s towards*

$$\frac{(H^{(m)}_{\mu_c,\nu})\,'(\frac{1}{\theta_j})}{\theta_j H^{(m)}_{\mu_c,\nu}(\frac{1}{\theta_j})} = \frac{1-c\int\frac{x^2}{(\theta_j-x)^2}d\nu(x)}{1+c\int\frac{x}{(\theta_j-x)}d\nu(x)}.$$

(ii) *for any spiked eigenvalue θ_l of A_N such that $\theta_l\neq\theta_j$, the norm of the orthogonal projection of $\xi_q(j)$ onto Ker $(\theta_l I_N - A_N)$ converges almost surely towards zero when N goes to infinity.*

For general deformed models, that is dealing with other matrices than Wigner matrices in the additive case or other matrices than Wishart matrices in the multiplicative case, such an exact separation phenomenon is not expected in full generality.

Nevertheless, I conjecture that

-the limiting values of the eigenvalues that separate from the bulk

-the limiting values of the norm of the orthogonal projection of the corresponding eigenvectors onto those associated to the spikes of the perturbation

will be given by the same quantities but involving the subordination functions relative to the limiting spectral distribution of the non-deformed model.

- one can check that this is true for instance by rewritting the results of F.Benaych-Georges-R. Rao concerning finite rank additive or multiplicative perturbation of a unitarily invariant matrix.

Subordination property in free probability definitely sheds light on spiked deformed models.