Martingales with continuous time and Brownian motion

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- Let $N^\infty = \cap_{p<\infty} L_p(N)$ the algebra of elements with finite moments of all order ($\|x\|_p = [\tau(|x|^p)]^{1/p}$);
- We shall also consider selfadjoint martingales, i.e. $b_t \in N_t^\infty$ such that $E_t(b_s) = b_t$ for $s \leq t$.

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- For many purposes we can work with a weaker vanishing variation condition: Let $(x_t) \subset N^\infty$. There exists a $p > 2$ such that for every $T > 0$

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Working definition: An abstract brownian motion is given by a selfadjoint martingale with almost uniformly continuous path such that $(b_t^2 - t)$ is again a martingale.

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2) The operator $\varphi_t(x) = \lim_{h\to 0} \frac{1}{h} E_t((b_{t+h} - b_t)x(b_{t+h} - b_t))$ is completely positive unital and trace preserving.

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3) The moments are given by

$$\tau(b_t^k) \ = \ \sum_{1\leq j<l\leq k} \int_0^t \tau(b_s^{j-1}\varphi_s(b_s^{k-j-1}))ds \ .$$

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More generally, one has a formula for joint moments.

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\begin{aligned}
&\tau((b_t - b_s)(b_t - b_s)(b_t - b_s)(b_t - b_s)) \\
&= \int_s^t \tau(db_r db_r (b_r - b_s)^2) + \int_s^t \tau(db_r (b_r - b_s) db_r (b_r - b_s)) \\
&\quad + \int_s^t \tau(db_r (b_r - b_s)^2 db_r) + \int_s^t \tau((b_r - b_s) db_r^2 (b_r - b_s)) \\
&\quad + \int_s^t \tau((b_r - b_s) db_r (b_r - b_s) db_r) + \int_s^t \tau((b_r - b_s)^2 db_r db_r) \\
&= 4 \int_s^t \tau((b_r - b_s)^2) dr + 2 \int_s^r \tau(db_r (b_r - b_s) db_r (b_r - b_s)) \\
&\leq 6 \int_s^t (r - s) dr \; = \; 3(t - s)^2 .
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Then one can construct an abstract brownian motion in a finite von Neumann algebra.

Bistochastic integral version of Doobs formula

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Theorem: There exists an abstract brownian motion (b_t) and an adapted process $a(s)$ such that

$$m_t \;=\; m_0 + \int_0^t a(s)^* db_s a(s) \, .$$

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iv) $b_r = b_r(x)$ is a twisted free brownian motion, i.e.

$$\tau(b_t^k) \;=\; \sum_{1\le j<l\le k} \int_0^t \tau(b_r^{j-1}\alpha_r(E_N(\alpha_{r^{-1}}(b_r^{l-j}))b_r^{k-j+1})dr\,.$$

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- These dilation results generalize the work of Kümmerer and Maassen in the matrix algebra situation.

Tools to distinguish brownian motion: Then notion of copies

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- Let $M \subset A$. We say that $\pi_j : A \to \mathcal{N}$ are copies over M if all the π_j are trace preserving *-homomorphism and $\pi_j|M = \pi_k|_M$.

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- Example $A = M \otimes N$ and $\mathcal{N} = M \otimes \otimes_{n\in\mathbb{N}} N$. Then we may use $\pi_j(m \otimes n) = m \otimes 1 \otimes \cdots \otimes n \otimes \cdots$, where n appears on the j-th position.

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- $M \subset A$ and $\mathcal{N} = *_M A$ is the infinite free product, π_j the *-homomorphism given by the j-coordinate.
- We say that copies $A_j = \pi_j(A)$ are subsymmetric (spreadable) if

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holds whenever $(j_1, ..., j_k)$ and $(l_1, ..., l_k)$ have the same order structure, i.e.

$$j_a < j_b \quad \Leftrightarrow l_a < l_b \, .$$

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holds for every permutation σ.
- Example: Let G be a group and $\alpha : G \to Aut(G)$ the group representation given by $\alpha_g(h) = ghg^{-1}$. Then one can define a group law on $G_\infty = \bigcup_n G^n$ such that the maps $\pi_j(g) = (1, ..., g, 1, ...,)$ give examples of subsymmetric, in general not symmetric copies.

Symmetry and subsymmetry

- We say that copies $A_j = \pi_j(A)$ are symmetric (exchangeable) if
$$\tau(\pi_{j_1}(a_1)\cdots\pi_{j_k}(a_k)) \;=\; \tau(\pi_{j_{\sigma(1)}}(a_1)\cdots\pi_{j_{\sigma(n)}}(a_k))$$
holds for every permutation σ.
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- **Remark:** By a famous Theorem of Aldous subsymmetric tight random variables are automatically symmetric.

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See Köstler's talk form more info on braid group and symmetry in particular

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Here stationary means that

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- For symmetric π_j, (ε_j) it suffices to consider pair partitions.

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gives a matrix model for the q-gaussian random variables.

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- Let $A_j = \pi_j(A)$ be subsymmetric copies, successively independent copies over $\mathbb{C}$ and ε_j subsymmetric. Let $x \in A$ with $\tau(x) = 0$ and $\tau(x^2) = 1$. Then the limit object
$$b_t \; = \; (\sqrt{t/n} \sum_{j=1}^{[tn]} \varepsilon_j \otimes \pi_j(x))^\bullet$$
is an abstract brownian motion.

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- In the Central Limit Model we can use the filtration given by $((N_{[tn]})^{\bullet})$. The continuous path condition follows from the moment formula (only sum of pair partitions).
- **Theorem:** There exists an abstract brownian motions with non-symmetric, independent increments: Indeed, we can find a group and a mean 0 element x such that the brownian motions constructed from the central limit theorem satisfies

$$\tau(g_1^3 g_2 g_1 g_2) \neq \tau(g_2^3 g_1 g_2 g_1) \,.$$

for $g_1 = B_2 - B_1$ and $g_2 = (B_3 - B_2)$.

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- Examples $f(\sigma) = q^{\mathrm{inversions}(\sigma)}$, but many more examples through the work Maassen and Guta.
- For $H = L_2(0,\infty)$ we may call $b_t = s(1_{[0,t]})$ a combinatorial brownian motion.

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exists. Moreover, for every γ, and $|I_l| = t$ we have

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- Moreover (J.+A.) $\Gamma_b(L_2(0,\infty))$ coincides with the von Neumann algebra generated by b_t's.

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- These results are related but not based on the results of Guta and Maassen on generalized Fock space constructions by species.

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Thanks for listening