On Free Product von Neumann Algebras

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Free Product von Neumann Algebra

$$\begin{matrix} (M_1, \varphi_1) \\ (M_2, \varphi_2) \end{matrix} \quad \rightsquigarrow \quad (M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$$

- The construction is fundamental in free probability theory, because it enables us to construct any number of freely independent random variables.
- The construction is also important in theory of von Neumann algebras, since it is one typical way to construct a new von Neumann algebra from known (given) ones.

Review on von Neumann Algebras 1/3

- $\mathcal{Z}(M) := \{x \in M \,|, \forall y \in M, xy = yx\} \cong L^\infty(\Omega, \nu)$ – center
- $M = \int_\Omega^\oplus M_\omega \nu(d\omega)$ – central decomposition
- Call M a factor if $\mathcal{Z}(M) = \mathbb{C}$.
- (Murray–von Neumann, 30's) The factors are classified into the classes:
  - type I_n ($\cong M_n(\mathbb{C})$),
  - type I_∞ ($\cong B(\mathfrak{H})$),
  - type II_1 ($\exists$ faithful normal tracial state),
  - type II_∞ ($\cong$ (type II_1) $\otimes B(\mathfrak{H})$),
  - type III (otherwise)

- $(M,\varphi) \rightsquigarrow \sigma_t^\varphi \in \mathbf{Aut}(M)$ – modular automorphism group
- Connes in ’73 further classified, by looking at σ_t^φ “modulo $\mathbf{Int}(M)$”, the type III into the following subclasses:
  - type III_0
  - type III_λ ($0 < \lambda < 1$)
  - type III_1
- $T(M) = \{t \in \mathbb{R} \,|\, \sigma_t^\varphi \in \mathbf{Int}(M)\}$ – T-set
- T-set IS NOT ENOUGH to determine III_λ-type in general; but
  - $T(M) = \{\frac{2\pi n}{-\log\lambda} \,|\, n \in \mathbb{Z}\}$ if M is a factor of type III_λ,
  - $T(M) = \{0\}$ if M is a factor of type III_1.

Review on von Neumann Algebras 3/3

Let's take a free ultrafilter $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$, a kind of 'extra-ordinary' natural numbers'

- M^ω – ultraproduct (a kind of "compactification" of M)
- $M_\omega \subset M' \cap M^\omega = \{x \in M^\omega \mid \forall y \in M, xy = yx\}$
  – asymptotic centralizer / a kind of "ideal boundary"
  (Remark: $M_\omega \neq M' \cap M^\omega$ in general; e.g. $M =$ hyperfinite III_λ factor.)
- Connes' observation: When M is a factor,

$$\mathbf{Int}(M) \text{ is closed if and only if } M_\omega = \mathbb{C}$$

- NB: hyperfiniteness or amenablity implies $M_\omega \neq \mathbb{C}$.

Fundamental Questions on $(M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$

- Determine the center and the central decomposition (factoriality question).
- Determine the Murray–von Neumann–Connes type (type classification question).
- Determine $M_\omega \subset M' \cap M^\omega$ (fullness question).
- Calculate further algebraic invariants – Sd- and τ- invariants, etc.
- Clarify the dependency of 'algebraic type' of M on the choice of φ_1, φ_2.

Previous Contributors/Partial Answers

- Barnett – factoriality, type classification, fullness
- Dykema – factoriality, type classification, Sd-invariant
- U – factoriality, type classification, (fullness, implicitly)
- —
- Radulescu – analysis on a special free product
- Shlyakhtenko – free Araki–Woods factors, τ-invariant
- Vaes – τ-invariant

Main Theorems 1/2

Theorem [U, arXiv:1011.5017]

Any free product $(M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$ with $\dim(M_i) \geq 2$ $(i = 1, 2)$, $\dim(M_1) + \dim(M_2) \geq 5$ is of the form $M = M_d \oplus M_c$, where:

- M_d is a multimatrix algebra possibly with $M_d = 0$,
- M_c is a factor of type II_1 or III_λ with $\lambda \neq 0$ with

$$T(M_c) = \{t \in \mathbb{R} \,|\, \sigma_t^{\varphi_1} = \mathrm{Id} = \sigma_t^{\varphi_2}\},$$

- and moreover $M_{c\,\omega} = M_c' \cap M_c^\omega = \mathbb{C}$.

Remarks:

- The multimatrix part M_d can be described explicitly.
- $(M_c)_\varphi$ (= the centralizer of $\varphi|_{M_c}$) can be small in general !
- Neither type I_∞, II_∞ nor III_0 occurs !

Main Theorems 2/2

Theorem [U, arXiv:1101.4991]

Let $(M,\varphi) = (M_1,\varphi_1) \star (M_2,\varphi_2)$ be a free product with $\dim(M_i) \geq 2$ $(i = 1,2)$, $\dim(M_1) + \dim(M_2) \geq 5$.

- If both φ_1, φ_2 are almost periodic, then so is φ, the Sd-invariant $\mathrm{Sd}(M_c)$ is the subgroup of $\mathbb{R}_+^\times$ algebraically generated by those λ such that $\sigma_t^{\varphi_i}(x) = \lambda^{it}x$ for some non-zero $x \in M_i$ with $i = 1,2$, and moreover $((M_c)_\varphi)' \cap M_c^\omega = \mathbb{C}$.
- The τ-invariant $\tau(M_c)$ is the weakest topology on $\mathbb{R}$ making both $t \in \mathbb{R} \mapsto \sigma_t^{\varphi_i} \in \mathrm{Aut}(M_i)$, $i = 1,2$, continuous.

Corollaries

Let $(M, \varphi) = (M_1, \varphi_1) \star (M_2, \varphi_2)$ with $\dim(M_i) \geq 2$ $(i = 1, 2)$.

- M is amenable if and only if $\dim(M_1) = \dim(M_2) = 2$.

Moreover, when $\dim(M_1) + \dim(M_2) \geq 5$ we have:

- The diffuse factor part M_c is always prime. (Chifan–Houdayer)
- If M (or M_c) has CMAP, then any non-amenable subalgebra of the diffuse factor part M_c has no Cartan. (Houdayer–Ricard)
- The diffuse factor part M_c has an almost periodic state if and only if both φ_1, φ_2 are almost periodic.

Analysis in the Diffuse Case 1/2

Lemma

If A is a diffuse von Neumann subalgebra of $(M_1)_\psi$ for some ψ (which may be different from the given φ_1) and if $x \in M$ satisfies $xAx^* \subseteq M_1$, then x must be in M_1.

Corollary

Any diffuse (semi-regular or singular) MASA A in M_1 which is the range of a faithful normal conditional expectation becomes again a (resp. semi-regular or singular) MASA in M.

Proposition 1

If either M_1 or M_2 is diffuse, then M is a factor of type II_1 or III_λ with $\lambda \neq 0$ and $T(M) = \{t \in \mathbb{R} \mid \sigma_t^{\varphi_1} = \mathrm{Id} = \sigma_t^{\varphi_2}\}$.

Analysis in the Diffuse Case 2/2

Lemma

If there exist ψ on M_1 (which may be different from φ_1) and $u, v \in ((M_1)_\psi)^u$ such that $\varphi_1(u^n) = \delta_{n0} = \psi(v^n)$, then

$$\|y(x - E^\omega(x))\|_{(\psi\circ E_1^\omega)} \leq \|[x, y]\|_{(\psi\circ E_1^\omega)}$$

for any $x \in \{u, v\}' \cap M^\omega$ and any $y \in \mathrm{Ker}(\varphi_2)$.

Proposition 2

If either M_1 or M_2 is diffuse, then $M' \cap M^\omega = \mathbb{C}$.

Next Problems

- Identify M_c with a free Araki–Woods factor when both M_1, M_2 are amenable.
  (A partial positive solution is known due to Houdayer.)
- Prove that M_c has no Cartan without any assumption.
  (This is true when M (or M_c) has CMAP due to Houdayer–Ricard.)

Thank you for your attention !