Exercises in **Advanced Probability Theory**

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In all Exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space.

For Exercises 1–3, let $A_n \in \mathcal{A}$ $(n = 1, 2, 3, ...)$.

1.) [3 points] Show that $\quad \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mathbb{P}(A_n)\,.$

2.) [3 points] State and prove a chain of inequalities between the quantities

$$\liminf \mathbb{P}(A_n)\,, \quad \limsup \mathbb{P}(A_n)\,, \quad \mathbb{P}\big(\liminf A_n\big)\,, \quad \mathbb{P}\big(\limsup A_n\big)$$

3.) [3 points] Prove the *Borel–Cantelli Lemma*:

$$\text{If} \quad \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \quad \text{then} \quad \mathbb{P}\big(\limsup A_n\big) = 0\,.$$

Conditional probability. For $A, B \in \mathcal{A}$, the conditional probability of A given B is

$$\mathbb{P}(A \mid B) = \begin{cases} \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}\,, & \mathbb{P}(B) > 0\,, \\ 0\,, & \mathbb{P}(B) = 0\,. \end{cases}$$

4.) [2 points] Let I be a finite or countable index set, and $B_i \in \mathcal{A}$ such that the B_i are pairwise disjoint and $\Omega = \bigcup_{i \in I} B_i$. Prove that for any $A \in \mathcal{A}$,

$$\mathbb{P}(A) = \sum_{i \in I} \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i).$$

(Rule of total probability)

5.) [2 points] For $A, B \in \mathcal{A}$, express $\mathbb{P}(B \mid A)$ in terms of $\mathbb{P}(A \mid B)$, that is, determine the correction factor in

$$\mathbb{P}(B \mid A) = \mathbb{P}(A \mid B) \times \text{correction factor}.$$

(Bayes' formula).

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6.) [3 points] Urn I contains 3 red and 7 white balls. Urn II contains 6 red and 3 white balls. We perform the following experiment: we first draw a random ball from urn I and transfer it to urn II. Then we draw a random ball from urn II.

(a) What is the probability that the ball drawn from urn II is red?

(b) Suppose that from urn II, we have drawn a white ball. Given this information, what is the probability that the ball transferred from urn I to urn II was also white?

Independence. $A, B \in \mathcal{A}$ are called independent, if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$
[so that $\mathbb{P}(A \mid B) = \mathbb{P}(A)$ when $\mathbb{P}(B) > 0$.]

$A_1, \ldots, A_n \in \mathcal{A}$ ($n \geq 3$) are called independent if for all choices of indices $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $k \geq 2$,

$$\mathbb{P}\big(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\big) = \mathbb{P}(A_{i_1})\,\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k}).$$

7.) [3 points] For $A \in \mathcal{A}$, write $A^1 = A$ and $A^{-1} = A^c$ (complement). Show that $A_1, \ldots, A_n$ are independent if and only if for all choices of $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$,

$$\mathbb{P}\big(A_1^{\varepsilon_1} \cap A_2^{\varepsilon_2} \cap \ldots \cap A_n^{\varepsilon_n}\big) = \mathbb{P}(A_1^{\varepsilon_1})\,\mathbb{P}(A_2^{\varepsilon_2}) \cdots \mathbb{P}(A_n^{\varepsilon_n}).$$

8.) [2 points] An urn contains 4 balls. The first carries the number 1, the second carries the number 2, the third carries the number 3, and the fourth carries the number 123. A ball is extracted at random.

Let A_i be the event that the extracted ball carries the *digit* i, where $i = 1, 2, 3$.

Show that the A_i are pairwise independent (i.e., any two of them are independent). Are all three events independent?