Graz University of Technology

Institut für Mathematische Strukturtheorie

Exercises in **Advanced Probability Theory**

Prof. Wolfgang Woess, Winter semester 2014/15

In all exercises, $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space.

Two random variables X, Y are called *independent* if for any $A \in \sigma(X)$ and $B \in \sigma(Y)$, the events A and B are independent.

9.) [3 points] Let X, Y be two independent, integrable (real valued) random variables. Show that $X \cdot Y$ is integrable and

$$\mathbb{E}(X \cdot Y) = \mathbb{E}(X)\,\mathbb{E}(Y).$$

Hints: use the steps of the costruction of the integral in the sense of Lebesgue. Recall from the lectures the specific way how a non-negative random variable can be approximated by a monotone sequence of simple ones.

10.) [3 points] Let X be an a.s. finite (real) random variable. Show that X is integrable if and only if

$$\sum_{k=1}^{\infty} k\,\mathbb{P}[k < |X| \le k+1] = \sum_{n=1}^{\infty} \mathbb{P}[|X| > n] < \infty\,.$$

(The exercise also comprises the proof of equality of the two sums.)

11.) [3 points] Prove the "independent version" of the Borel-Cantelli Lemma:

If $(A_n)_{n\ge 1}$ is a sequence of *independent* events in $\mathcal{A}$ then

$$\sum_n \mathbb{P}(A_n) < \infty \iff \mathbb{P}\big(\limsup A_n\big) = 0$$
$$\sum_n \mathbb{P}(A_n) = \infty \iff \mathbb{P}\big(\limsup A_n\big) = 1$$

Hint: for proving the "missing" part, pass to the complements, and use the inequality $1+x \le e^x$ for real x.

Recall that a *semi-ring* is a collection $\mathcal{S}$ of subsets of Ω such that

- $\emptyset \in \mathcal{S}$
- $A, B \in \mathcal{S} \implies A \cap B \in \mathcal{S}$
- $A, B \in \mathcal{S} \implies \exists\ C_1 \ldots, C_n \in \mathcal{S}: \quad A \setminus B = \biguplus_{i=1}^{n} C_i\,.$

Our $\mathcal{S}$ will have the addtional property that Ω is a countable union of elements in $\mathcal{S}$.

A *ring* $\mathcal{R}$ is a semiring with the additional property that $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$.

Recall that a *finitely additive* (finite) *measure* on a semiring is a mapping $\mathbf{m} : \mathcal{S} \to [0, \infty)$ such that $\mathbf{m}(\emptyset) = 0$ and

$$\mathbf{m}\left(\biguplus_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} \mathbf{m}(C_i) \quad \text{whenever} \quad C_i \in \mathcal{S} \quad \underline{\text{and}} \quad \biguplus_{i=1}^{n} C_i\,.$$

It is called σ-additive, if the same is also valid for countable disjoint unions belonging to $\mathcal{S}$.

The following exercise should be known from Measure Theory.

12.) [3 points] Let $\mathbf{m}$ be a finitely additive (finite) measure on a semiring $\mathcal{S}$. Show that the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$ consists of the collection of all finite, disjoint unions of sets in $\mathcal{S}$, and construct the unique extension of $\mathbf{m}$ to a finitely additive measure on that ring.

13.) [3 points] Use Exercise 12 (same assumptions!) to show the following:
let $(A_n)_{n \ge 1}$ be a sequence of pairwise disjoint elements of $\mathcal{S}$ whose union also belongs to $\mathcal{S}$. Then

$$\mathbf{m}\left(\biguplus_{n=1}^{\infty} A_n\right) \ge \sum_{n=1}^{\infty} \mathbf{m}(A_n).$$

For Exercises 14–15, let

$$\mathcal{S} = \{(a\,,\,b] : a, b \in \mathbb{R}\,,\ a \le b\}$$

and $F : \mathbb{R} \to [0, 1]$ be a *distribution function* (monotone increasing, continuous from the right, with limits 0 at $-\infty$ and 1 at $+\infty$).

14.) [2 points] Verify that $\mathcal{S}$ is a semi-ring and that

$$\mathbf{m}_F\big((a\,,\,b]\big) = F(b) - F(a)$$

defines a finitely additive measure on $\mathcal{S}$.

15.) [4 points] Show that $\mathbf{m}_F$ is σ-additive on $\mathcal{S}$.

Hints: you need to consider the situation when

$$(a\,,\,b] = \biguplus_{n} (a_n\,,\,b_n]$$

Use Exercise 13.), and show by contradiction that the inequality cannot be strict. I.e., assume the difference of right and left hand side is $\varepsilon > 0$. You can replace $(a\,,\,b]$ by a smaller interval $[d\,,\,b]$ and enlarge $(a_n\,,\,b_n]$ to $(a_n\,,\,d_n)$. The numbers d and d_n have to be chosen suitably, using the properties of F.

Note: Exercise 15 and the extension machinery from measure theory imply that there is a unique probability measure on $(\mathbb{R}, \mathcal{B})$ which extends $\mathbf{m}_F$.