Graz University of Technology

Exercises in **Advanced Probability Theory**

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16.) [3 points] Let X be a random variable whose distribution function $F(x) = \mathbb{P}[X \le x]$ is continuous. Determine the distribution of the composed random variable $F(X)$.

17.) [3 points] Let $F(x)$ be a continuous distribution function on $\mathbb{R}$ and U a (continuous) uniform random variable on the interval $[0,1]$ (open or closed or half-open does not matter!). Find a real function $G(x)$ such that the distribution function of the composed random variable $X = G(U)$ is $F(x)$.

Hint for 16 & 17: use a suitable "partial inverse function" of F. (Start first with strictly increasing F and then think about the case when F has flat pieces.)

18.) [3 points] Let $(\Omega, \mathcal{A}, \mathbb{P}) = \big([0\,,\,1), \mathcal{B}_{[0,1)}, \lambda\big)$, where λ is Lebesgue measure. Define a sequence of random variables X_n $(n \in \mathbb{N})$ on this space by

$$X_n(\omega) = \begin{cases} 0\,, & \lfloor 2^n \omega \rfloor \text{ is even,} \\ 1\,, & \lfloor 2^n \omega \rfloor \text{ is odd} \end{cases}$$

Determine the distribution of X_n and show that these random variables are independent.

19.) [3 points] Let $(X_n)_{n\ge1}$ be a sequence of i.i.d. Bernoulli $B(1,\frac{1}{2})$ random variables (fair coin tosses). Determine the distribution of

$$\sum_{n=1}^{\infty} 2^{-n}\, X_n\,.$$

Hint: the distribution of a given function of a sequence of i.i.d. random variables (X_n) does not depend on the concrete realisation of those random variables. That is, you may choose your favorite model for the underlying probability space and construction of the random variables. (It does not have to be a product space.)

Additional hint: there is a reason why this exercise comes right after the previous one! (Recall what it means that a real number is written, for example, in decimal digit expansion.)

There is also a reason why the next exercise comes right after the preceding two exercises!

20.) [3 points] Provide a construction of a sequence $(U_n)_{n\ge1}$ of i.i.d. random variables defined on the Lebesgue probability space of exercise 18 which have (continuous) uniform distribution on the unit interval.

21.) [3 points] Combine the preceding exercises for the following. Let $F(x)$ be a continuous distribution function on $\mathbb{R}$. Provide a construction of a sequence $(X_n)_{n\ge1}$ of i.i.d. random variables defined on the Lebesgue probability space of exercise 18, each of which has distribution function F.