22.) [3 points] Let $(\Omega, \mathcal{A}, \mathbb{P}) = \big([0\,,\,1), \mathcal{B}_{[0,1)}, \lambda\big)$ be the Lebesgue probability space, and let

$$(\Omega^*, \mathcal{A}^*, \mathbb{P}^*) = \big(\{0,1\}, \mathcal{P}(\{0,1\}), B(1,\tfrac{1}{2})\big)^{\mathbb{N}}$$

be the product space of countably many copies of the standard Bernoulli space (i.e., $B(1,\frac{1}{2})$ is Bernoulli distribution with $\theta = \frac{1}{2}$).
Consider the mapping $\tau : [0\,,\,1) \to \Omega^*$,

$$\tau(\omega) = \big(X_n(\omega)\big)_{n\in\mathbb{N}}\,,$$

where X_n is as in Exercise 18.
Show that τ is measurable and injective. Determine $\mathbb{P}^*\big(\Omega^* \setminus \tau(\Omega)\big)$. Show that τ is measure preserving, that is, $\mathbb{P}(\tau^{-1}A) = \mathbb{P}^*(A)$ for every $A \in \mathcal{A}^*$.

23.) [3 points] Let $T : [0\,,\,1) \to [0\,,\,1)$ be the mapping $T(\omega) = 2\omega - \lfloor 2\omega \rfloor$ on the Lebesgue probability space.
Prove that T is measure preserving and ergodic.
Hint: you can combine Exercise 22 with the Kolmogorov 0-1 Law.

24.) Let $(X_n)_{n\ge1}$ be a sequence of independent, real random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

a) [3 points] Show that the σ-algebras $\mathcal{A}(X_1,\dots,X_k)$ and $\mathcal{A}(X_{k+1},\dots,X_n)$ are independent for any choice of k, n with $1 \le k < n$.
Hint: recall that $\mathcal{A}(X_{k+1},\dots,X_n)$ consists of all events $[(X_1,\dots,X_k) \in B]$, where B is in the Borel σ-algebra of $\mathbb{R}^k$. Use tha fact that the latter is generated by the semialgebra of all sets $B_1 \times \dots B_k$, where B_j is a Borel set in $\mathbb{R}$, and analogously for $\mathbb{R}^{n-k}$.

b) [3 points] Show that the σ-albegras $\mathcal{A}(X_1,\dots,X_k)$ and $\mathcal{A}(X_{k+1},X_{k+2},\dots)$ are independent.

Deduce that if $f(X_1,\dots,X_k)$ and $g(X_{k+1},X_{k+2},\dots)$ are measurable (real) functions of the respective sequences, then they are also independent.
Remark: “measurable” means that $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, and that $g : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is well defined and Borel measurable with respect to the infinite product Borel σ-algebra restricted to the range of $(X_{k+1},X_{k+2},\dots)$.
Example: $|X_n| \le 1$ for all n, and $g(X_{k+1},X_{k+2},\dots) = \sum_{n=k+1}^{\infty} 2^{-n}\,X_n$.

c) [2pt] Now let $(Y_n)_{n\ge1}$ be a second sequence of independent random variables, independent of $(X_n)_{n\ge1}$. [That is, the random variables $X_1, Y_1, X_2, Y_2, \dots$ are all independent.]
Show that $\mathcal{A}(X_1,X_2,\dots)$ and $\mathcal{A}(Y_1,Y_2,\dots)$ are independent.

Deduce that, as an example, if $|X_n|, |Y_n| \le 1$ for all n, then

$$\sum_{n=1}^{\infty} 2^{-n}\,X_n \quad\text{and}\quad \sum_{n=1}^{\infty} 2^{-n}\,Y_n \quad\text{are independent.}$$

For a complex-valued random variable $U + \mathfrak{i}\,V$, the expectation is defined as $\mathbb{E}(U + \mathfrak{i}\,V) = \mathbb{E}(U) + \mathfrak{i}\,\mathbb{E}(V)$, if both U and V are integrable ($\iff \mathbb{E}(|U + \mathfrak{i}\,V|) < \infty$).

The *characteristic function* $\varphi_X : \mathbb{R} \to \mathbb{C}$ of a real random variable X is defined as

$$\varphi_X(t) = \mathbb{E}(e^{\mathfrak{i}\,tX})\,.$$

25.) [3 points]

a) Determine the characterisitc function of X, if X has Bernoulli distribution $B(1, \theta)$

b) Let X, Y be two independent random variables. Express the characteristic function of $X + Y$ in terms of the characteristic functions of X and of Y.

c) Determine the characteristic function of X, if X has binomial distribution $B(n, \theta)$.

Use parts a and b for solving c!

26.) [3 points] Determine the characteristic function of X, if X is

a) Poisson distributed with parameter $\xi > 0$,

b) uniformly distributed on $[0\,,\,1]$,

c) exponentially distributed with parameter $\lambda > 0$.