
31.) [2 points] Let $(\Omega, \mathcal{A}, \mathbb{P}) = \big((0\,,\,1], \mathcal{B}_{(0,1]}, \lambda\big)$, where λ is Lebesgue measure. Let $\mathcal{F}$ be the σ-algebra generated by the atoms $C_i = (2^{-i}\,,\,2^{-i+1}]$, $i \in \mathbb{N}$. Compute $\mathbb{E}(X \mid \mathcal{F})$ for the simple random variable

$$X(\omega) = \begin{cases} -1\,, & \omega \in (0\,,\,1/3] \\ 0\,, & \omega \in (1/3\,,\,2/3] \\ 1\,, & \omega \in (2/3\,,\,1]. \end{cases}$$

32.) [3 points] Let $\Omega = [-1\,,\,1]$ with the Borel σ-algebra and $\mathbb{P} = \frac{1}{2}\,\lambda$, where λ is Lebesgue measure. Consider the σ-algebra $\mathcal{F}$ of all Borel sets A which are symmetric, that is, $A = -A$, where $-A = \{-x : x \in A\}$.
Given an arbitrary integrable real random variable X on Ω, find an explicit formula for $\mathbb{E}(X \mid \mathcal{F})$.

The first part of the following exercise serves to further refresh the memory on conditional expectation with respect to atomic σ-algebras.

33.) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $\mathcal{Y}$ be a countable set, equipped with the σ-algebra of all subsets. We consider an $\mathcal{Y}$-valued stochastic process, that is, a sequence of $\mathcal{Y}$-valued random variables $(Y_n)_{n\ge 1}$.
Let

$$p_n(y_1, \dots, y_n) = \mathbb{P}[Y_1 = y_1, \dots, Y_n = y_n]\,, \quad y_1, \dots, y_n \in \mathcal{Y},$$

be the joint distribution of $(Y_1, \dots, Y_n)$.

(a) [1 point] Now let $X : \Omega \to \mathbb{R}$ be a *discrete* random variable. Write out the formula for

$$\mathbb{E}(X \mid Y_1, \dots, Y_{n-1}).$$

(b) [1 point] In particular, let $X_n = g_n(Y_1, \dots, Y_n)$, where $g_n : \mathcal{Y}^n \to \mathbb{R}$. Write out the formula for

$$\mathbb{E}(X_n \mid Y_1, \dots, Y_{n-1}).$$

(c) [1 point] Write out what it means that the sequence of RVs (X_n) of (b) is a martingale with respect to (Y_n), that is, with respect to the sequence of σ-algebras $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

34.) [3 points] Let P be a probability measure on the integers $\mathbb{Z}$ with finite first moment $\sum |k|\, P(k)$, and let Y_n be i.i.d. random variables with distribution P.
For which functions $g : \mathbb{Z} \to \mathbb{R}$ is

$$X_n = g(Y_1 + \cdots + Y_n)$$

a martingale with respect to (Y_n) ?

35.) [3 points] With notation as in exercise 32, let (Y_n) be a time-homogeneous Markov chain with initial distribution $\nu(y) = \mathbb{P}[Y_1 = y]$ and transition probabilities

$$p(y, z) = \mathbb{P}[Y_n = z \mid Y_{n-1} = y],$$

so that

$$\mathbb{P}[Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n] = \nu(y_1)p(y_1, y_2) \cdots p(y_{n-1}, y_n).$$

For which functions $g : \mathcal{Y} \to \mathbb{R}$ is

$$X_n = g(Y_n)$$

a martingale with respect to (Y_n) ?