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36.) Let Y be a random variable with values in $\mathbb{N}_0$, and P its distribution; $P(n) = \mathbb{P}[Y = n]$. The *probability generating function* (p.g.f.) of Y, resp. P, is

$$f_Y(z) = f_P(z) = \sum_{n=0}^{\infty} P(n)\, z^n\,, \quad |z| \le 1\,.$$

(a) [2 points] Compute the left-sided derivative $f_P'(1-)$. Verify that 1 is a fixed point of $f_P(z)$. Discuss the shape of that function on $[0\,,\,1]$. Under which conditions is there another fixed point besides 1 ?

(b) [2 points] Given two independent random variables Y_1, Y_2 with values in $\mathbb{N}_0$, express the p.g.f. of $Y_1 + Y_2$ in terms of f_{Y_1} and f_{Y_2}.

Let P be a probability distribution on $\mathbb{N}_0$. The *Galton-Watson process with offspring distribution* P is a stochastic process $(X_n)_{n\ge 0}$ evolving on $\mathbb{N}_0$ as follows.

$X_0 = 1$ and, if $X_{n-1} = k$ then the distribution of X_n is that of the sum k i.i.d. random variables each with distribution P. That is,

$$\mathbb{P}[X_n = m \mid X_{n-1} = k] = \mathbb{P}[Y_1^{(n)} + \cdots + Y_k^{(n)} = m]\,,$$

where $\big(Y_i^{(n)}\big)$ is a double sequence of i.i.d. P-distributed RVs.

Interpretation: "family tree" starting with 1 ancestor in generation 0. X_n is the number of elements in the n-th generation. Each member in each generation produces a random number of offspring (children) independently of all other family members, with distribution P.

We are interested in the *extinction probability* $\lambda = \mathbb{P}[\exists \text{ n: } X_n = 0]$.

We suppose that we do *not* have $P(0) + P(1) = 1$.

37.) Let $g_n(z)$ be the p.g.f. of X_n.

(a) [2 points] Use the preceding exercise and $f_P(z)$ to show that $g_n = f_P \circ g_{n-1}$. Apply this to relate $\mathbb{P}[X_n = 0]$ with $\mathbb{P}[X_{n-1} = 0]$.

(b) [2 points] Justify that $\mathbb{P}[X_n = 0] \nearrow \lambda$.
Deduce from this fact and (a) that λ is the smallest fixed point of $f_P(z)$ in $[0\,,\,1]$.

38.) (a) [2 points] Give a general criterion for $\mathbb{P}[\text{survival}] = 1 - \lambda > 0$.

(b) [1 point] Compute λ when $P(0) = q$, $P(2) = p$, where $p + q = 1$.

39.) Note that (X_n) is a Markov chain: conditionally upon X_n, the random variables X_{n+1} and $(X_1, \dots, X_{n-1})$ are independent.

(a) [3 points] Use this fact to compute

$$\mathbb{E}(X_{n+1}|X_n, \dots X_1) = \mathbb{E}(X_{n+1} \mid \mathcal{F}_n)\,,$$

where $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

(b) [2 points] Find a sequence of positive numbers M_n such that $(X_n/M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$.
Is the martingale limit theorem applicable ?