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Mathematical foundations of information theory
SS 2018
Exercise Sheet 1

14^{th} of March 2018

Exercise 1 (1 Point)**.** Is $\{\emptyset, \{b\}, \{a, c\}, \{d, b\}, \{a, b, c, d\}\}$ a σ-algebra over $\Omega = \{a, b, c, d\}$? If not, which elements are missing? How would you interprete the "information" described by the correct σ-algebra?

Exercise 2 (1 Point)**.** Let $A, B \subset \Omega$ be events with $\mathbb{P}(B) > 0$. Show that:

(a) $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) \iff \mathbb{P}(A \,|\, B) = \mathbb{P}(A)$

(b) Let $A \cap B = \emptyset$. Show that $\mathbb{P}(A \,|\, B) = 0$.

(c) $\mathbb{P}(A \,|\, \Omega) = \mathbb{P}(A)$. What does that mean?

Exercise 3 (1 Point)**.** Let $A, B \subset \Omega$ be independent events. Show that:

(a) The events A and $B^c = \Omega \setminus B$ are independent.

(b) The events A^c and B^c are independent.

Exercise 4 (2 Points)**.** Consider two independent rolls of a fair six-sided die, and the following events: $A = \{\,\text{1st roll is}\, 1, 2 \,\text{or}\, 3\}$, $B = \{\,\text{1st roll is}\, 3, 4 \,\text{or}\, 5\}$ and $C = \{\,\text{the sum of the two rolls is}\, 9\}$.

(a) Show that $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C)$.

(b) Are the three events independent?

Exercise 5 (2 Points)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $A, B, C \in \mathcal{A}$ some events. Show that:

(a) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

(b) $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) + \mathbb{P}(A \cap B \cap C)$.

(c) $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$. When do we have equality?

(d) (Rule of total probability) Let $B \subseteq \biguplus_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{A}$ for each $n \in \mathbb{N}$. Then

$$\mathbb{P}(B) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)\,\mathbb{P}(B \,|\, A_n)\,.$$

Exercise 6 (2 Point)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $A \in \mathcal{A}$ and $(A_n)_{n \in \mathbb{N}}$ be a sequence of sets, which fulfils $A_n \in \mathcal{A}$ and $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$. Moreover it holds

$$\lim_{n \to \infty} A_n = A\,.$$

Show that $\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A)$.