Adrian Fuchs and Gundelinde Wiegel

Mathematical foundations of information theory & Discrete Stochastics and Information Theory SS 2018

Exercise Sheet 11

6^{th} of June 2018

Exercise 44 (1 Point)**.** Let $X \in \{0, 1, 2\}$ be the input, and $Y \in \{0, 1, 2\}$ the output of a channel with transition matrix (note: $p(y|x) := \mathbb{P}(Y = y|X = x)$)

$$Q = \begin{bmatrix} p(0|0) & p(1|0) & p(2|0) \\ p(0|1) & p(1|1) & p(2|1) \\ p(0|2) & p(1|2) & p(2|2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} .$$

Show that the channel capacity is $C = \log_2[3] - 1$.

Exercise 45 (2 Points)**.** Let the random variable X be the input, and the random vector $Y = (Y_1, Y_2)$ the output of a channel. The two components of Y are conditionally independent and conditionally identically distributed given X. Show that:

$$I(X; Y_1, Y_2) = 2\, I(X; Y_1) \;-\; I(Y_1; Y_2)\,,$$

and conclude that the capacity of the channel is less then twice the capacity of the channel with input X and output Y_1.

Exercise 46 (2 Points)**.** We consider two independent discrete channels $(\mathcal{X}_1, \mathcal{P}_1, \mathcal{Y}_1)$ and $(\mathcal{X}_2, \mathcal{P}_2, \mathcal{Y}_2)$. We can construct a new channel such that $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ will be transmitted in parallel (at the same time): so x_1 is mapped to some $y_1 \in \mathcal{Y}_1$ and x_2 is mapped to some $y_2 \in \mathcal{Y}_2$. Express the capacity of this joint channel in terms of the two individual capacities.

Exercise 47 (2 Points)**.** Let Σ be an alphabet with $|\Sigma| = 2^n$. The codebook $x^{(n)}$ maps uniquely each element of the alphabet to a binary sequence of length n. This code will be sent over the n-th extension of a binary symmetric channel with crossover probability p_n, i.e., $\mathbb{P}[X = 1, Y = 0] = \mathbb{P}[X = 0, Y = 1] = p_n$. Then it will be decoded by $g := \left(x^{(n)}\right)^{-1}$. Find conditions on $(p_n)_{n\in\mathbb{N}}$ so that the average error probability goes to zero as n goes to infinity.