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Mathematical foundations of information theory & Discrete Stochastics and Information Theory
SS 2018
Exercise Sheet 12

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Exercise 48 (2 Points)**.** The elements of the alphabet $\mathcal{W} := \{a, b, c, d\}$ are uniformly distributed and will be decoded with the following code:

$$C(a) = 00,\ C(b) = 01,\ C(c) = 10,\ C(d) = 11\,.$$

The coded elements of $\mathcal{X}$ will be transmitted through a discrete channel. Hereby the following error can occur: with probability 0.1 a 1 instead of a 0 is transmitted and with probability 0.05 a 0 instead of a 1 is transmitted. The received bit-pairs will be decoded according to C. The error during the transmission occurs independently for each bit.

(a) Calculate the maximal error probability $\lambda^{(2)}$, and the average error probability $p_{err}^{(2)}$.

(b) Suppose that both 0 and 1 are transmitted wrongly with the same probability p_0. What is the maximum of p_0 such that the maximal error probability is below 5%?

Exercise 49 (2 Points)**.** Let $X_1, X_2, \ldots, X_n$ be a sequence of binary input random variables and $Y_1, Y_2, \ldots, Y_n$ a sequence of binary output random variables. We assume that $Y_1, Y_2, \ldots, Y_n$ is conditionally independent given $X_1, X_2, \ldots, X_n$, that is, Y_i is conditionally independent of everything else given X_i for each $i = 1, \ldots, n$. We define $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$ with the conditional distribution given by $p(y, x) = \Pi_{i=1}^n p_i(y_i|x_i)$ and

$$\begin{bmatrix} p_i(0|0) & p_i(1|0) \\ p_i(0|1) & p_i(1|1) \end{bmatrix} = \begin{bmatrix} 1-p_i & p_i \\ p_i & 1-p_i \end{bmatrix}.$$

Find $max_{p(x)} I(X;Y)$.

Exercise 50 (2 Points)**.** Show that the capacity of the channel with probability transition matrix

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is achieved by a distribution that places zero probability on one of the input symbols. Calculate the capacity of the channel. Give an intuitive reason why that letter is not used.

Exercise 51 (2 Points)**.** Let (X_i, Y_i, Z_i) be i.i.d. according to $p(x, y, z)$. We will say that $(\underline{x}, \underline{y}, \underline{z})$ is jointly typical (i.e. $(\underline{x}, \underline{y}, \underline{z}) \in A_\varepsilon^{(n)}$) if they are pairwise jointly typical and $p(\underline{x}, \underline{y}, \underline{z}) \in 2^{-n(H(X,Y,Z) \pm \varepsilon}$.

Suppose that $(\underline{\tilde{X}}, \underline{\tilde{Y}}, \underline{\tilde{Z}})$ is distributed according to $p(\underline{x})p(\underline{y})p(\underline{z})$. Thus $\underline{\tilde{X}}, \underline{\tilde{Y}}, \underline{\tilde{Z}}$ have the same marginals as $p(\underline{x}, \underline{y}, \underline{z})$ but are independent. Find bounds on $\mathbb{P}[\underline{\tilde{X}}, \underline{\tilde{Y}}, \underline{\tilde{Z}} \in A_\varepsilon^{(n)}]$ in terms of entropies $H(X), H(Y), H(Z), H(X,Y), H(X,Z), H(Y,Z)$ and $H(X,Y,Z)$.