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Mathematical foundations of information theory
SS 2018
Exercise Sheet 2

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Exercise 7 (1 Point). The joint distribution of the vector (X, Y) is given by the following matrix:

(i,j)	-1	2	3
0	0.03	0.16	0.12
1	0.07	0.35	0.27

For example, $\mathbb{P}[(X,Y) = (0,-1)] = 0.03$.

a) Calculate the marginal distributions of X and Y.

b) Are X and Y independent?

Exercise 8 (2 Points). We consider a model with two urns: Urn I contains 3 red and 7 white balls and Urn II contains 7 red and 2 white balls. Now we proceed in two steps, namely
Step I: Draw randomly a ball from Urn I and put it into Urn II,
Step II: Draw randomly a ball from Urn II independent of the drawing before.

(a) What is the probability, that the ball drawn in Step II is red?

(b) What is the probability, that the drawn ball from Urn I was white, given that the ball drawn in Step II was red? (Hint: Use Bayes Theorem.)

Exercise 9 (2 Points). Show that a random variable X is almost surely constant, if and only if

$$\mathbf{Var}(X) = 0\,.$$

(A random variable X is called almost surely constant, if $\mathbb{P}[X = a] = 1$.)

Exercise 10 (2 Points). For a nonnegative integer-valued random variable X, show that

$$\sum_{i=0}^{\infty} i\,\mathbb{P}[X > i] \;=\; \frac{1}{2}(\mathbf{E}(X^2) \;-\; \mathbf{E}(X))\,.$$

Exercise 11 (1 Point). If $\mathbf{E}(X) = 1$ and $\mathbf{Var}(X) = 5$ find

(a) $\mathbf{E}(2 + X^2)$

(b) $\mathbf{Var}(4 + 3\,X)$

Exercise 12 (2 Points). Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $\mathbb{P}[X_i = 0] = p$ and $\mathbb{P}[X_i = 1] = 1-p$ for all $i \in \mathbb{N}$. Show that for $p \in [0,1)$

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i^2} = 1 \text{ almost surely.}$$